PRIMES 2026 – Entrance Problem Set

October 15, 2025

Notation: We let $\mathbb{N}$, $\mathbb{N}_0$, $\mathbb{Z}$, and $\mathbb{R}$ denote the sets of positive integers, nonnegative integers, integers, and real numbers, respectively.

Problem 1. [Analysis] Let p be a positive real. Prove that there exists $c_p \in \mathbb{R}$ such that if f is a continuously differentiable real-valued function on $[-1, 1]$ with $f(1) > f(-1)$ and $|f'(s)| \leq 1$ for all $s \in (-1, 1)$, then there exists $s_0 \in (-1, 1)$ with $f'(s_0) > 0$ such that

$$|f(s) - f(s_0)| \leq c_p |s - s_0|^{\frac{1}{p}} \quad \text{for all} \quad s \in (-1, 1).$$

Recommended reading for analysis:

- *Basic Analysis I & II: Introduction to Real Analysis by J. Lebl (Chapters 1–5)*

Problem 2. [Abstract Algebra] Fix an odd prime p, and let $\mathbb{F}_p$ be the field of p elements. Let G be the group of upper-unitriangular 3×3 matrices over $\mathbb{F}_p$ via

$$(x, y, z) \longleftrightarrow \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}.$$

(a) Prove that the commutator of G is the center Z of G, and compute the conjugacy classes of G.

(b) Show that the maximal abelian subgroups of G are precisely the subgroups of order p^2 containing Z, and they are in one-to-one correspondence with the 1-dimensional subspaces (lines) of $G/Z \cong \mathbb{F}_p^2$.

(c) Find the group of automorphisms of G in terms of $\mathbb{F}_p$ and $GL_2(\mathbb{F}_p)$.

Recommended reading for algebra:

- *Algebra: Abstract and Concrete by F. M. Goodman (Chapters 1–6)*

Problem 3. [Combinatorics] For $n \in \mathbb{N}$, call a function $f : [n] \to [n]$ *prime-tail* if, in each weakly connected component of its functional digraph, the unique directed cycle has *prime* length, and each vertex on the cycle has at most one incoming edge from outside the cycle. Let a_n be the number of prime-tail mappings on $[n]$. Determine a *closed form* for the generating function $A(x)$ of the sequence $(a_n)_{n \geq 1}$ and extract a coefficient formula for a_n. Show that $A(x)$ can be written as a single exponential built from cycle lengths and labeled lists, and then derive a nontrivial asymptotic for a_n as $n \to \infty$.

Recommended reading for combinatorics:

- *Enumerative Combinatorics, vol. 1, by R. Stanley (Chapters 1–3)*

Problem 4. [Probability] Let S be a circle in $\mathbb{R}^2$ and assume that b blue points and r red points have been chosen uniformly and independently at random on S for some $b, r \in \mathbb{N}$ with $b, r \geq 3$. Let P be the intersection of the convex hull of the red points and the convex hull of the blue points, and let p be the number of vertices of P (in particular, $p = 0$ when P is empty). Find the expected value of p.

Recommended reading for probability:

- MIT Course 18.05 Introduction to Probability and Statistics (Chapters 15–19)

Problem 5. [Number Theory] Fix an integer $n \geq 2$ and choose an integer x_0 with $\gcd(x_0, n) = 1$. Define a sequence $(x_k)_{k\geq 0}$ as follows:

$$x_{k+1} = x_k + \gcd(x_k, n) \qquad (k \geq 0).$$

Let $T(n, x_0)$ be the least $t \geq 0$ such that $n \mid x_t$.

(a) Prove that $T(n, x_0)$ is finite for every such x_0.

(b) Write $n = \prod_{i=1}^{r} p_i^{a_i}$ with distinct primes p_i and $a_i \geq 1$. Show that

$$T(n, x_0) \ \leq \ \sum_{i=1}^{r} a_i \,(p_i - 1).$$

(c) Prove that equality holds for all prime powers $n = p^a$ and, more generally, for n prime. Give at least one explicit choice of x_0 achieving equality in those cases.

Recommended reading for number theory:

- *A Classical Introduction to Modern Number Theory by K. Ireland and M. Rosen (Chapters 1–4)*

Problem 6. [Linear Algebra] Fix $k \in \mathbb{N}$ and, for each $v \in \mathbb{R}^k$, let $||v||$ denote the Euclidean norm in $\mathbb{R}^k$. Then define $||M||_k := \sup_{||v||\leq 1} ||Mv||$ for any $k \times k$ complex matrix M. Prove that $||M^k|| \leq \frac{k}{\ln 2}||M||^{k-1}$ for all complex matrix M provided that all the eigenvalues of M have absolute value at most 1.

Recommended reading for linear algebra:

- *Linear Algebra As an Introduction to Abstract Mathematics by I. Lankham, B. Nachtergaele and A. Schilling (Chapters 3–7)*

Problem 7. [Algorithms] Let $\mathscr{P}_n$ consist of all subsets of $\{0, 1, \dots, n\}$ containing 0, and let $\mathscr{A}_n$ be the set of all $A \in \mathscr{P}_n$ with $|A| \geq 2$ such that for any $B, C \in \mathscr{P}_n$ the equality $A = \{b + c : (b, c) \in B \times C\}$ implies that $1 \in \{|B|, |C|\}$. For any $k, n \in \mathbb{N}$ with $1 \leq k \leq n$, set

$$\mathscr{A}_{n,k} := \big\{A \in \mathscr{A}_n : |A| = k\big\}.$$

(a) Design an efficient algorithm to compute $v_n := (|\mathscr{A}_{n,1}|, \dots, |\mathscr{A}_{n,n}|)$ given an input $n \in \mathbb{N}$.

(b) Produce a list with all the vectors $v_1, v_2, \dots, v_{35}$ and provide a code that we can use to reproduce your 35 vectors.

(c) Is it true that for any $n \in \mathbb{N}$, we can pick $k \in \mathbb{N}$ with $1 \leq k \leq n$ such that $|\mathscr{A}_{n,1}| \leq \dots \leq |\mathscr{A}_{n,k}|$ and $|\mathscr{A}_{n,k}| \geq \dots \geq |\mathscr{A}_{n,n}|$?

Recommended reading for algorithms:

- MIT Course 6.006 Introduction to Algorithms (Chapters 15–19)