Computing Isomorphisms Between Groups Defined by Integer Matrices

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Introduction

$A \in \mathrm{M}_n(\mathbb{Z})$ is nonsingular. Define $G_A = \{A^{-k}x : x \in \mathbb{Z}^n, k \in \mathbb{Z}_{\geq 0}\}$ where $\mathbb{Z}^n \subseteq G_A \subseteq \mathbb{Q}^n$.

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Such groups arise in the study of:

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The Isomorphism Problem: Given nonsingular $A, B \in M_n(\mathbb{Z})$, when is $G_A \cong G_B$?

Groups in Dimension One

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Definition

$N = \det A$. The *ring of N-adic rationals* is the set

$$\mathscr{R} = \mathscr{R}(A) = \mathbb{Z}\left[\frac{1}{N}\right] = \left\{ \frac{x}{N^k} \,\middle|\, x, k \in \mathbb{Z} \right\}.$$

Isomorphism Problem

Isomorphisms $T : G_A \to G_B$ are linear maps $T \in GL_n(\mathbb{Q})$ satisfying $T(G_A) = G_B$.

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$$A = \begin{pmatrix} 3 & 0 \\ 0 & 6 \end{pmatrix}, \qquad B = \begin{pmatrix} 9 & 8 \\ 0 & 12 \end{pmatrix}, \qquad C = \begin{pmatrix} 9 & 0 \\ 0 & 12 \end{pmatrix}.$$

$$G_A = G_C = \mathbb{Z}\left[\frac{1}{3}\right] \oplus \mathbb{Z}\left[\frac{1}{6}\right].$$

We have the isomorphism $T : G_B \to G_C$ where

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Isomorphic groups may correspond to matrices with different characteristic polynomials.

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with $G_A \cong G_B \cong G_C$. Next, consider

$$A' = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{pmatrix}, \quad B' = \begin{pmatrix} 4 & 2 & 6 \\ 0 & 9 & 8 \\ 0 & 0 & 12 \end{pmatrix}, \quad C' = \begin{pmatrix} 4 & 5 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 12 \end{pmatrix}.$$

$G_{A'} \ncong G_{B'}$. However, $G_{A'} \cong G_{C'}$ and

$$\mathrm{Iso}(G_{A'}, G_{C'}) = \left\{ \begin{pmatrix} a & b & 0 \\ 0 & b & 0 \\ c & d & e \end{pmatrix} : \begin{array}{l} a \in \mathbb{Z}\left[\frac{1}{2}\right]^{\times}, \ b \in \mathbb{Z}\left[\frac{1}{3}\right]^{\times}, \\ e \in \mathbb{Z}\left[\frac{1}{6}\right]^{\times}, \ c, d \in \mathbb{Z}\left[\frac{1}{6}\right] \end{array} \right\}.$$

Generalized Eigenspaces

Definition

A *generalized eigenvector* v of A satisfies $(A - \lambda I)^k v = 0$ for $k \in \mathbb{Z}_{\geq 0}$. Denote $X_{A,\mathfrak{p}} = \bigoplus_{\mathfrak{p}|\lambda} \ker(A - \lambda I)^n$ for a prime ideal $\mathfrak{p}$ in $\mathscr{O}_K$, where $K = \mathbb{Q}(\lambda)$ and $\mathscr{O}_K$ is the ring of integers of K.

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$$A = \begin{pmatrix} 4 & 5 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 12 \end{pmatrix}.$$

Then,

$$X_{A,2} = \langle \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}^t, \begin{pmatrix} 0 & 0 & 1 \end{pmatrix}^t \rangle$$

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If $T : G_A \to G_B$ is an isomorphism, then $T(X_{A,\mathfrak{p}}) = X_{B,\mathfrak{p}}$ is necessary for all $\mathfrak{p} \mid \det A$ where $\mathfrak{p} \subseteq \mathscr{O}_K$.

p-adic Localization

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Let $p \in \mathbb{N}$ be prime. The group

$$\overline{G}_{A,p} = G_A \otimes_{\mathbb{Z}} \mathbb{Z}_p = \left\{ A^{-k} x : x \in \mathbb{Z}_p^n, k \in \mathbb{Z}_{\geq 0} \right\}$$

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Definition

$\overline{G}_{A,p} \cong D_p(A) \oplus R_p(A)$, where $D_p(A) \cong \mathbb{Q}_p^{t_p(A)}$ and $R_p(A) \cong \mathbb{Z}_p^{n-t_p(A)}$ where $t_p(A)$ is the multiplicity of zero in h_A reduced modulo p. We call $D_p(A)$ the *divisible part* and $R_p(A)$ a *reduced part* of $\overline{G}_{A,p}$.

p-adic Localization

Example

Consider

$$A = \begin{pmatrix} 1 & 1 & 1 \\ -2 & 4 & 1 \\ -4 & 4 & 4 \end{pmatrix}$$

with characteristic polynomial $(x-2)(x-3)(x-4)$.

$\overline{G}_{A,2} \cong \mathbb{Q}_2^2 \oplus \mathbb{Z}_2$ and $\overline{G}_{A,3} \cong \mathbb{Q}_3 \oplus \mathbb{Z}_3^2$.

$\overline{G}_{A,p} \cong \mathbb{Z}_p^3$ for every prime $p \geq 5$.

Past Work

Theorem (Sabitova, 2026)

It is decidable whether groups G_A and G_B are isomorphic when $n = 2$ or when the characteristic polynomials of A and B are irreducible in $\mathbb{Z}[x]$.

Main Result

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2. **Reduce isomorphism candidates**: Reduce the set of possible isomorphisms using $T(X_{A,\mathfrak{p}}) = X_{B,\mathfrak{p}}$.

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3. **Apply p-adic conditions:** impose restrictions on T through p-adic analysis of characteristics.
4. **Establish computability:** $\mathrm{Iso}(G_A, G_B)$ can be computed in finite time.

Main Result

Example

$$A = \begin{pmatrix} 2 & 0 & 1 \\ 0 & 0 & -6 \\ 0 & 1 & 1 \end{pmatrix}, \qquad B = \begin{pmatrix} 2 & 1 & 1 \\ 0 & 0 & -3 \\ 0 & 2 & 1 \end{pmatrix}.$$

$A' = M^{-1}AM$ and $B' = N^{-1}BN$ are the Jordan forms of A and B. If $T : G_A \to G_B$ is an isomorphism, we may write $C = N^{-1}TM$.

$$A' = B' = \mathrm{diag}(2, \lambda, \overline{\lambda}) \qquad \lambda = \frac{1 + \sqrt{-23}}{2}.$$

We have the restriction $C = \mathrm{diag}(a, b, \overline{b})$ where $a, b \in K^\times$.

$$\mathrm{Iso}(G_A, G_B) = \left\{ NCM^{-1} : a = \pm 2^x, b = \pm \lambda^y (\lambda - 2)^z, x, y, z \in \mathbb{Z} \right\}.$$

Future Work

Consider parameterized families of these groups defined by monic characteristic polynomials.

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Case $n = 2$: characteristic polynomial $x^2 + x + c$ for $c \in \mathbb{Z}$ and companion matrices.

Theorem

$A, B \in M_2(\mathbb{Z})$ are companion matrices of characteristic polynomials $x^2 + x + m$ and $x^2 + x + n$, respectively.

$$G_A \cong G_B \iff m = n.$$

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