On Open Borel Orbits in Conormal Bundles to Schubert Cells

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Flag Varieties

- $G = \mathrm{GL}_n(\mathbb{C}) = \{\text{invertible } n \times n \text{ matrices over } \mathbb{C}\}$
- $B = \{\text{upper triangular elements of } G\}$
- G/B is called the *flag variety*
- Points $gB/B \in G/B$ correspond to *complete flags*, which are chains of subspaces such that $\dim V_i = i$:

$$\{0\} = V_0 \subset V_1 \subset \cdots \subset V_n = \mathbb{C}^n$$

- If $v_1, \ldots, v_n$ are the columns of g, this identification takes gB to the flag

$$\{0\} \subset \mathrm{span}\{v_1\} \subset \mathrm{span}\{v_1, v_2\} \subset \cdots \subset \mathrm{span}\{v_1, \ldots, v_n\}$$

Schubert Varieties

- B acts on G/B by left multiplication
- Via Gaussian elimination, we can write every B-orbit BgB/B in the form BwB/B, where w is a permutation matrix
- We call $C(w) := BwB/B$ a *Schubert cell*
- We call the closure $X(w) := \overline{C(w)}$ a *Schubert variety*

The Simplest Example

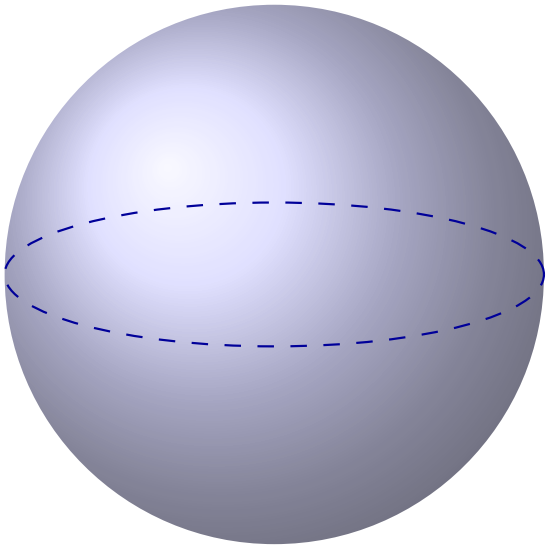
- Take $n = 2$
- Complete flags in $\mathbb{C}^2$ are just 1-dimensional subspaces:

$$\{0\} = V_0 \subset V_1 \subset V_2 = \mathbb{C}^2$$

- Thus, G/B looks like $\mathbb{CP}^1$, the space of lines through $(0,0)$ in $\mathbb{C}^2$
- The lines in $\mathbb{C}^2$ through $(0,0)$ are either of the form $\text{span}\{(a,1)\} \subseteq \mathbb{C}^2$ for $a \in \mathbb{C}$ or $\text{span}\{(1,0)\} \subseteq \mathbb{C}^2$
- Thus, G/B looks like $\mathbb{C}$ with a point at infinity, or S^2

The Simplest Example, II

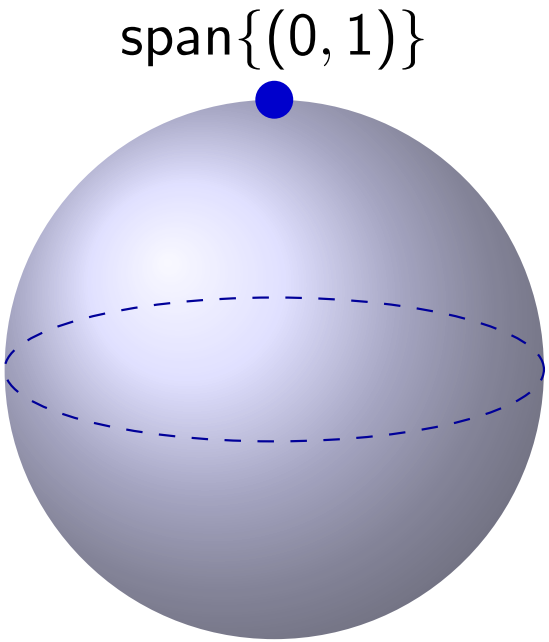
- We now describe the Schubert cells in $G/B \cong S^2$
- The span of $(1,0) \in \mathbb{C}^2$ is fixed by the B-action
- The rest of the 1-dimensional subspaces are transitively permuted by B



G/B

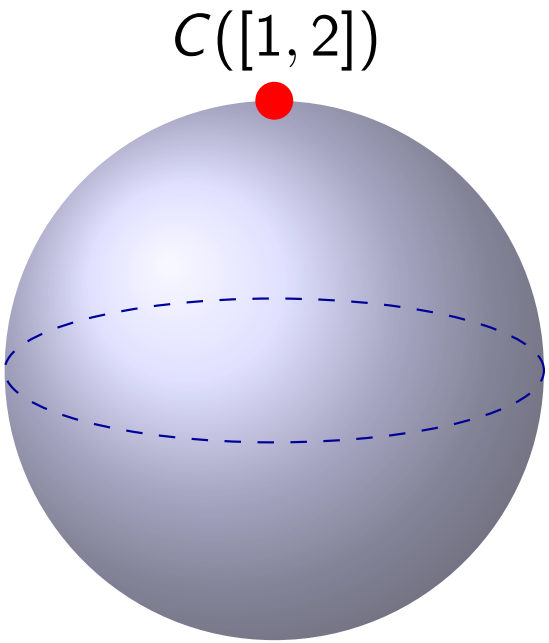
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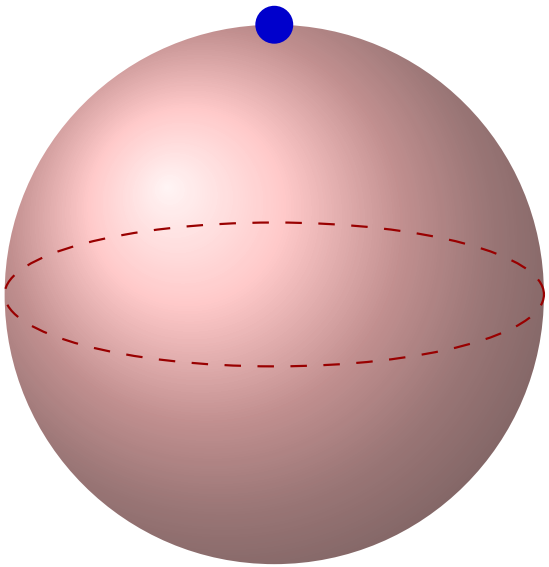
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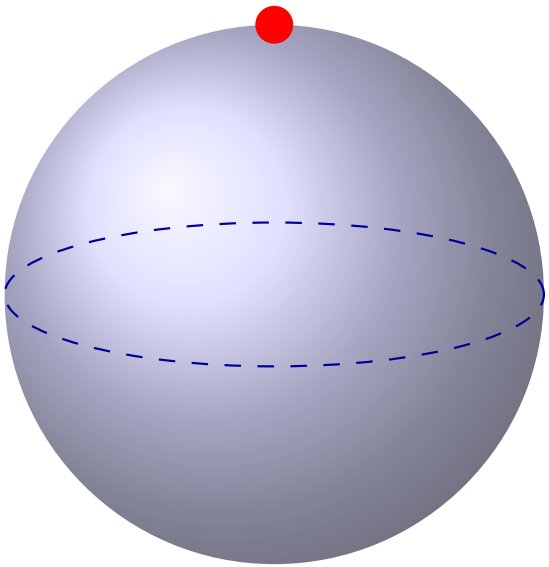


$C([2,1])$

The Simplest Example, II

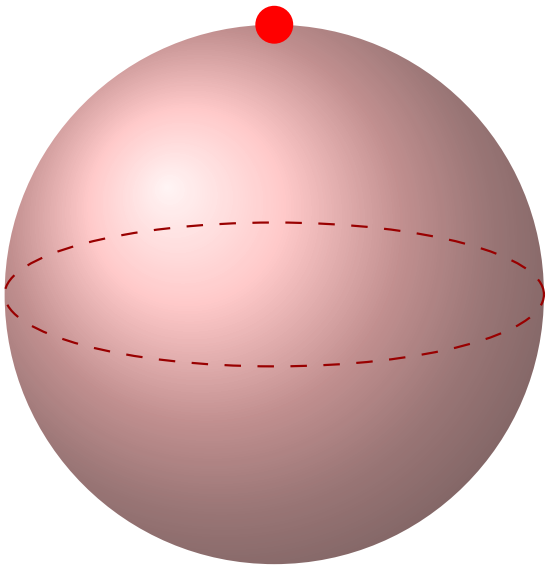
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$$C([1,2]) = X([1,2])$$



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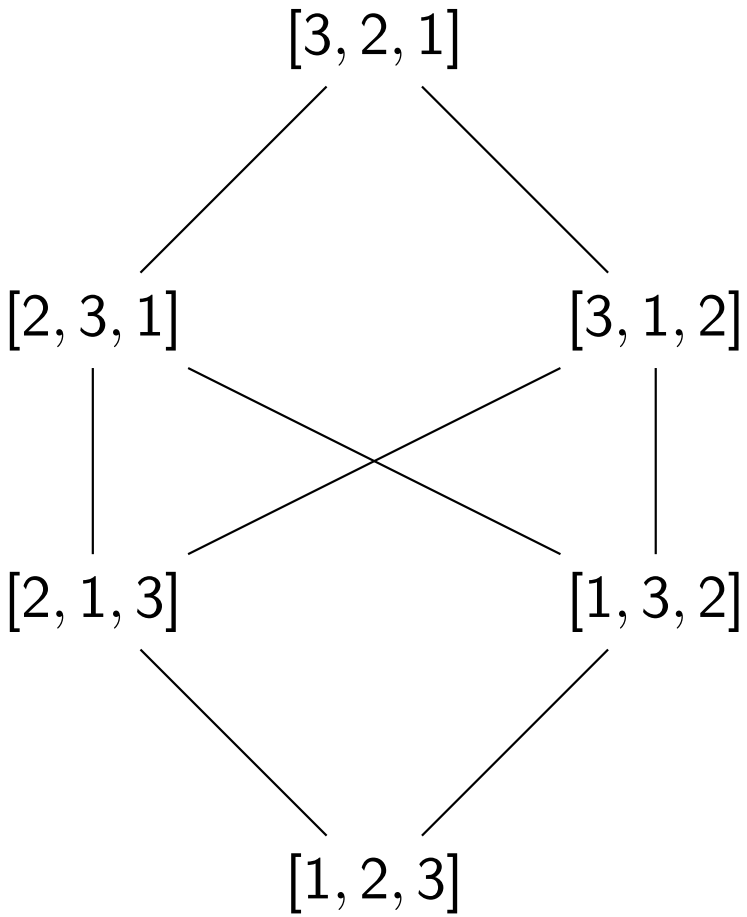
$$X([2,1]) = C([1,2]) \cup C([2,1])$$

Bruhat Order

- We define the Bruhat order $\leq$ such that $v \leq w$ iff $C(v) \subseteq X(w)$
- In the $n = 2$ case, we have $C([1,2]) \subseteq X([2,1])$, so $[1,2] \leq [2,1]$ is the Bruhat order on S_2
- The permutation w_0 such that $1 \mapsto n$, $2 \mapsto n-1$,... is maximal with respect to $\leq$
- In S_2, we have $w_0 = [2,1]$

The Second-Simplest Example

- Below is the Bruhat order on S_3



Why Should We Care?

- Geometric representation theory
  - Important for studying the representation theory of Lie(G)
  - Used to prove the Kazhdan–Lusztig conjectures about the Kazhdan–Lusztig polynomials $P_{v,w}$

The Motivating Theorem

- 231-avoidance is a simple combinatorial property of a permutation w

Theorem (Lapid–Minguez, [LM18])

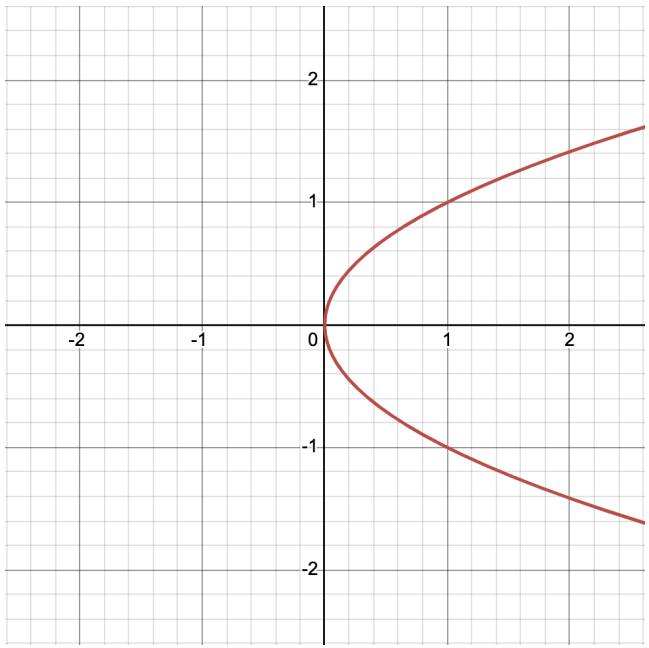
Let $v \leq w$ such that w is 231-avoiding. The following are equivalent:

1. $X(w_0 w)$ is smooth on $C(w_0 v)$.
2. The conormal bundle to $C(v) \subseteq X(w)$ contains an open B-orbit.

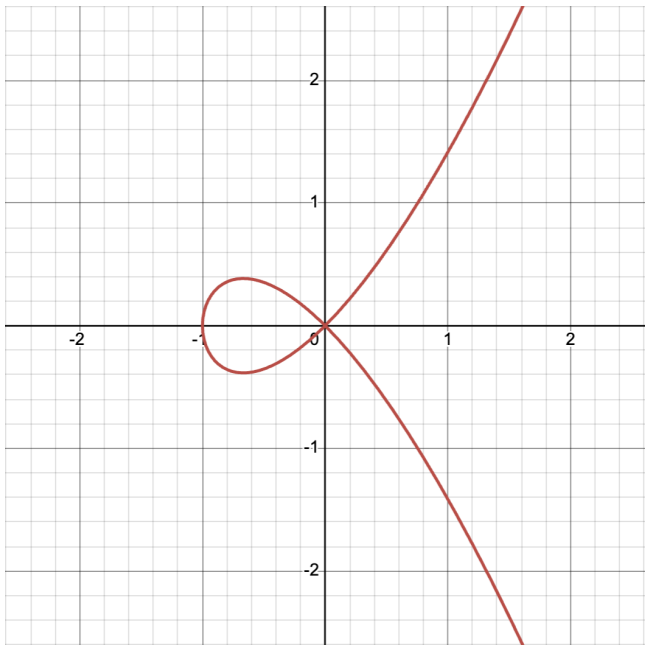
- Condition (1) is well-understood and intimately connected with representation theory
- Condition (2) is also closely related to representation theory, but is poorly understood
- Thus, understanding condition (2) in more generality than the 231-avoiding case is of general interest

Smoothness

- Smoothness means the tangent space has the same dimension as the space itself
- Consider the example of the space defined by $y^2 - x = 0$

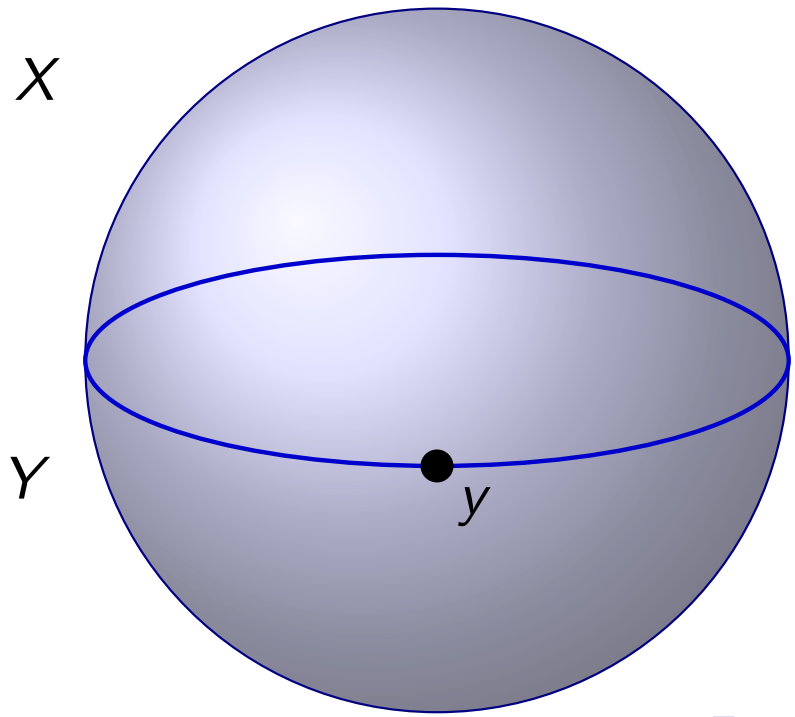


Smoothness, II

- Singularity (i.e., non-smoothness) means having "too many tangent directions"
- Let's look at the curve in $\mathbb{R}^2$ defined by $y^2 - x^2 - x^3 = 0$
- 

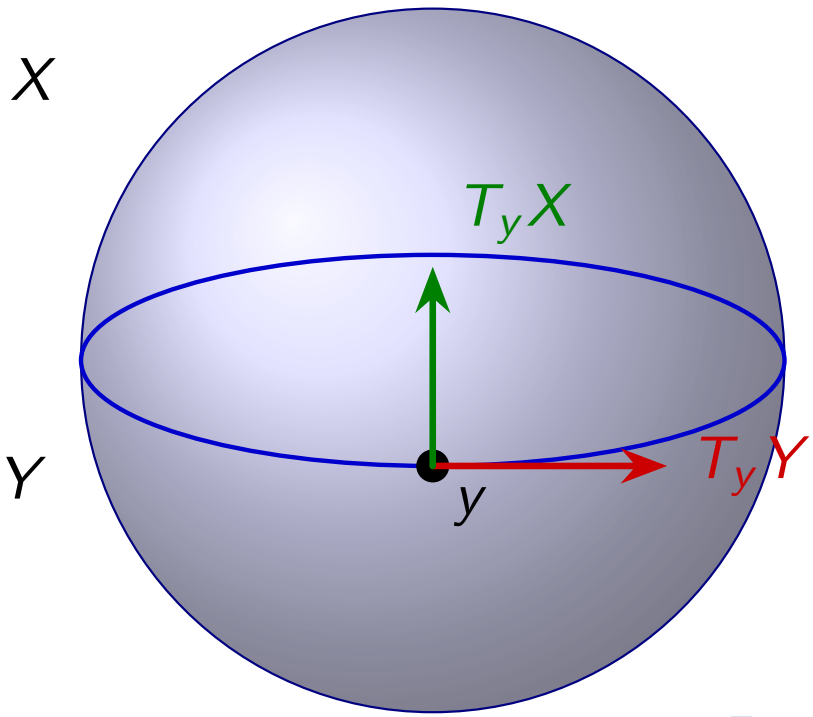
Conormal Bundle

- Take $Y \subseteq X$ and a point $y \in Y$



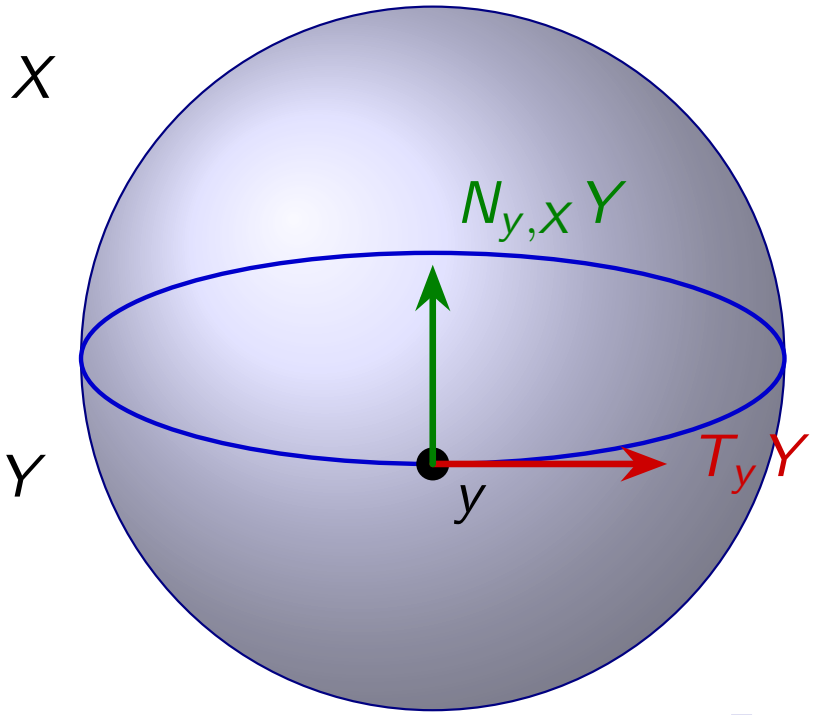
Conormal Bundle

- Take $Y \subseteq X$ and a point $y \in Y$
- Consider the tangent spaces $T_y X$ and $T_y Y$



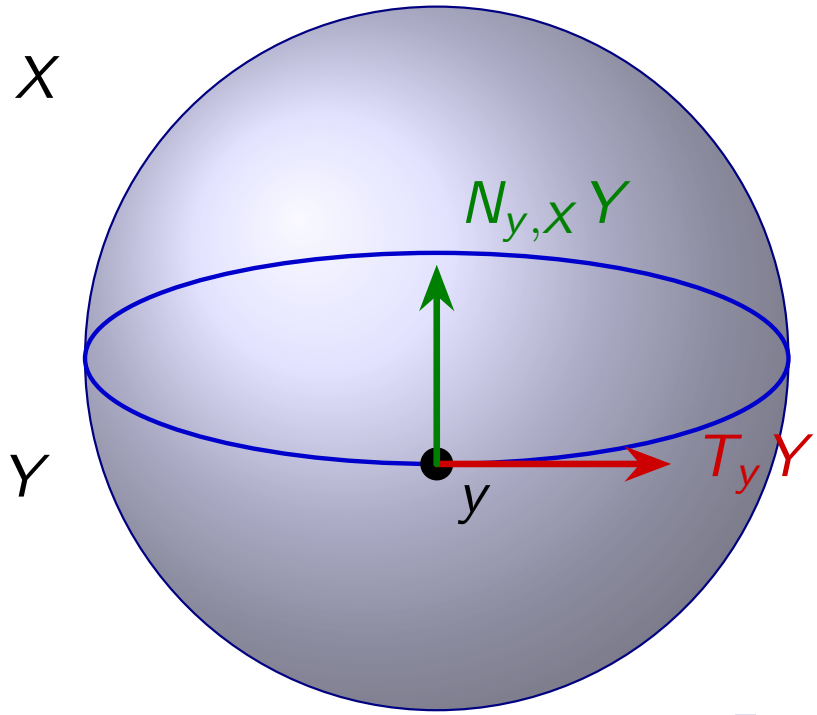
Conormal Bundle

- Take $Y \subseteq X$ and a point $y \in Y$
- Consider the tangent spaces $T_y X$ and $T_y Y$
- Consider the normal space $N_{y,X} Y$, the orthogonal complement of $T_y Y$ in $T_y X$



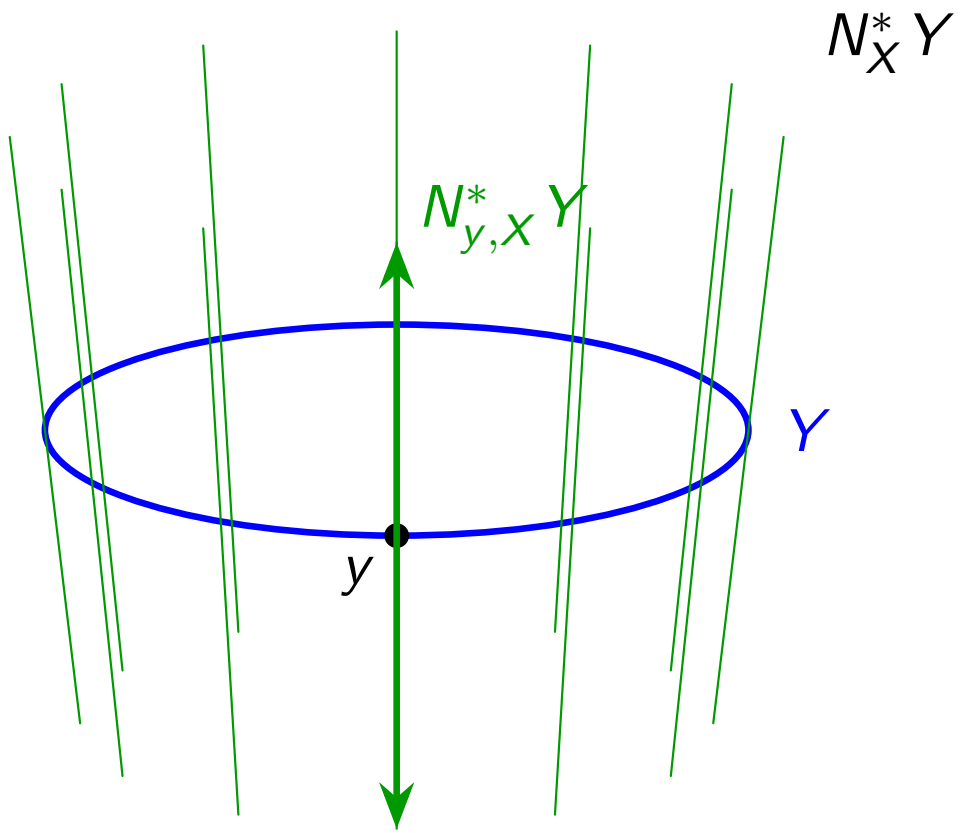
Conormal Bundle

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- Consider the tangent spaces $T_y X$ and $T_y Y$
- Consider the normal space $N_{y,X} Y$, the orthogonal complement of $T_y Y$ in $T_y X$
- We define the conormal space to $Y \subseteq X$ at y as the dual $(N_{y,X} Y)^*$



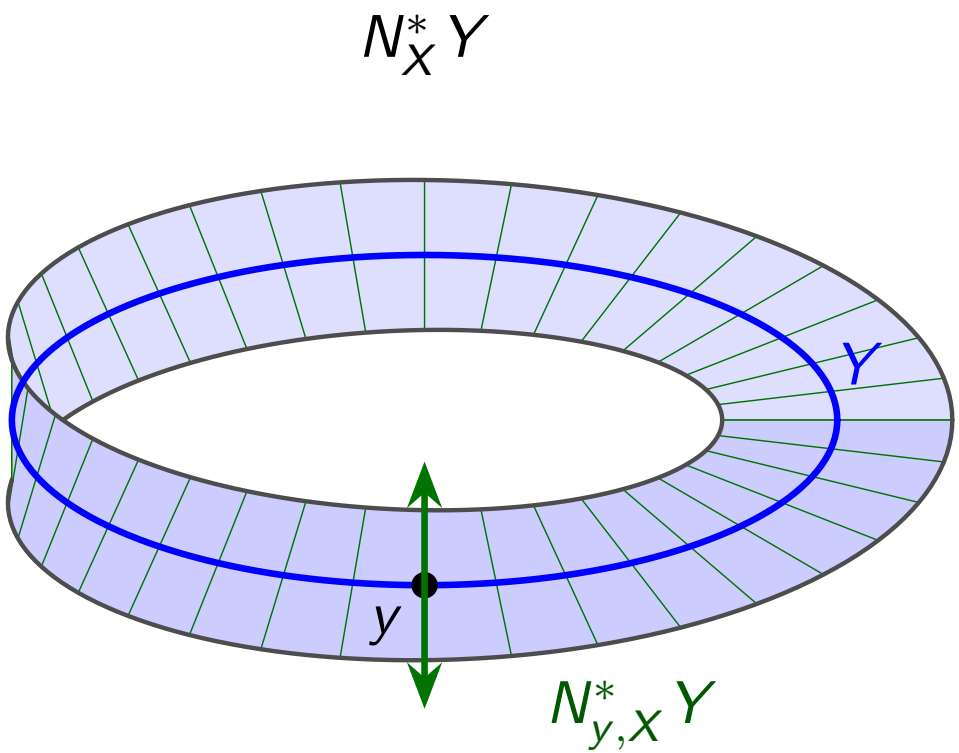
Conormal Bundle, II

- The conormal bundle to $Y \subseteq X$ comes from gluing the conormal space at each point to each point $y \in Y$



Conormal Bundle, III

- The fibers can glue together in nontrivial ways



Conormal Bundle, IV

- There is a natural B-action on the conormal bundle to $C(v) \subseteq X(w)$
- Roughly, B moves the points of $C(v)$, so it also moves the points of the conormal bundle
- We are interested in when this B-action has an open ("big") orbit

The Problem We Solve

Conjecture (Mellit, [LM25])

Let $v \leq w$ such that $X(w)$ is smooth. The following are equivalent:

1. $X(w_0 w)$ is smooth on $C(w_0 v)$.
2. The conormal bundle to $C(v) \subseteq X(w)$ contains an open B-orbit.

- If w avoids 231, then $X(w)$ is smooth
- Thus, Mellit's conjecture is strictly stronger than Lapid–Minguez's theorem
- This is very surprising!

Our Result

Theorem

Let $v \leq w$ such that $X(w)$ is smooth. The following are equivalent:

1. $X(w_0 w)$ is smooth on $C(w_0 v)$.
2. The conormal bundle to $C(v) \subseteq X(w)$ contains an open B-orbit.

- Strictly speaking, we use an even less general hypothesis than smoothness
- Thus, we settle a stronger version of Mellit's conjecture and vastly generalize Lapid–Minguez's theorem

Remarks on the Proof

- In fact, the proof that the existence of an open orbit implies smoothness does not use the smoothness hypothesis
- To prove our result, we combinatorially characterize the pairs $v \leq w$ such that there exists an open B-orbit
  - We construct graphs $\mathcal{G}_i(v, w)$ for $1 \leq i \leq n$
  - We show that there is a cycle in some $\mathcal{G}_i(v, w)$ iff there is no open B-orbit

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Thank you.

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