Semisimplification of the Coordinate Ring of $\mathrm{GL}(mJ_1 \oplus nJ_{p-1})$ in Positive Characteristic

Harry Gao
Mentor: Dr. Arun S. Kannan

MIT PRIMES Conference

October 25, 2026

Outline

- Representation Theory of GL(n)
- Super objects
- Semisimplification
- Main results
- Acknowledgements

What is a Group Action?

Definition (Group action)

A group G *acts* on a set X if each $g \in G$ moves each $x \in X$ to some $g \cdot x \in X$, such that:

- Identity: $e \cdot x = x$,
- Compatibility: $g \cdot (h \cdot x) = (gh) \cdot x$.

- There is an underlying map $f : G \times X \to X$ with $f(g, x) = g \cdot x$.

Example

- The dihedral group D_{2n} on a regular n-gon acts on the set of n vertices by permuting their positions.

What is a Representation?

Definition (Representation)

A representation is a group action $f : G \times X \to X$, where X is a vector space and f is linear in $x \in X$. Equivalently, f is a group homomorphism $G \to \mathrm{GL}(X)$.

Example

- S_3 acts on points $(x, y, z) \in \mathbb{R}^3$ by permuting coordinates.
- $\mathrm{GL}(V)$ acts on V by multiplication.

Representations of $\mathrm{GL}(n)$

- We are interested in representations $\rho : G \to \mathrm{GL}(W)$, where $G = \mathrm{GL}(n)$.
- So, ρ sends a matrix $g \in G$ to another matrix $\rho(g) \in \mathrm{GL}(W)$.
- These representations are *rational*: the entries of $\rho(g)$ are polynomials in the entries x_{ij} of g and $\det(g)^{-1}$, i.e. elements of the coordinate ring $\mathcal{O}(\mathrm{GL}(n)) = k[x_{ij}, \det^{-1}]$.
- The equivalent action map $G \times W \to W$ can be encoded by the map $W \to \mathcal{O}(\mathrm{GL}(n)) \otimes W$, so the structure of representations over $\mathrm{GL}(n)$ is encapsulated by $\mathcal{O}(\mathrm{GL}(n))$.

Super setting

Definition (Super vector space)

A super vector space is a $\mathbb{Z}_2$-graded vector space V which decomposes into $V = V_0 \oplus V_1$.

- A homogeneous vector $v \in V$ is one that lies entirely in V_0 or V_1.
- Its parity, $|v|$, is 0 if it is in V_0 (even) and 1 if it is in V_1 (odd).
- V has super dimension $m|n$ if the even part has dimension m and the odd part has dimension n.

- Unlike ordinary vector spaces, swapping homogeneous vectors v, w introduces a sign:

$$v \otimes w \mapsto (-1)^{|v||w|}\, w \otimes v.$$

This is the **Koszul sign rule**.

Super setting

Definition (GL($m|n$))

The general linear supergroup GL($m|n$) consists of $(m+n)$ by $(m+n)$ matrices:

$$\begin{array}{c} \\ 1,\ldots,m \\ \bar{1},\ldots,\bar{n} \end{array} \begin{array}{c} \begin{array}{cc} 1,\ldots,m & \bar{1},\ldots,\bar{n} \end{array} \\ \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] \end{array}$$

whose Berezinian $\mathrm{Ber}(M) = \det(A - BD^{-1}C)\det(D)^{-1}$ is invertible. Here A, D have even entries and B, C have odd entries.

- The overarching goal is to improve our understanding of the representation theory of GL($m|n$) in positive characteristic.
- This requires understanding $\mathcal{O}(\mathrm{GL}(m|n))$.
- **Semisimplification** provides a way to go from the classical setting, GL($m + n(p-1)$), to the super setting, GL($m|n$).

Semisimplification

- Work in characteristic p and consider a $\mathrm{Mat}(n)$ module M.
- Pick an element t of $\mathrm{Mat}(n)$ with $t^p = 0$, so that $t^p \cdot v = 0$ for all $v \in M$.
- Decompose M into Jordan blocks J_i with $i \leq p$:

$$J_i : v_1 \xrightarrow{t} t \cdot v_1 \xrightarrow{t} t^2 \cdot v_1 \xrightarrow{t} \cdots \xrightarrow{t} t^{i-1} v_1 \xrightarrow{t} 0$$

Definition (Semisimplification)

Semisimplification is a functor between categories which quotients by negligible morphisms.

Semisimplification

- J_1 is sent to L_1, J_{p-1} is sent to L_{p-1}, and J_p is sent to the zero object.
- For $p > 2$, L_1 and L_{p-1} generate a subcategory equivalent to sVec, with the L_1 forming an even basis and the L_{p-1} forming an odd basis.

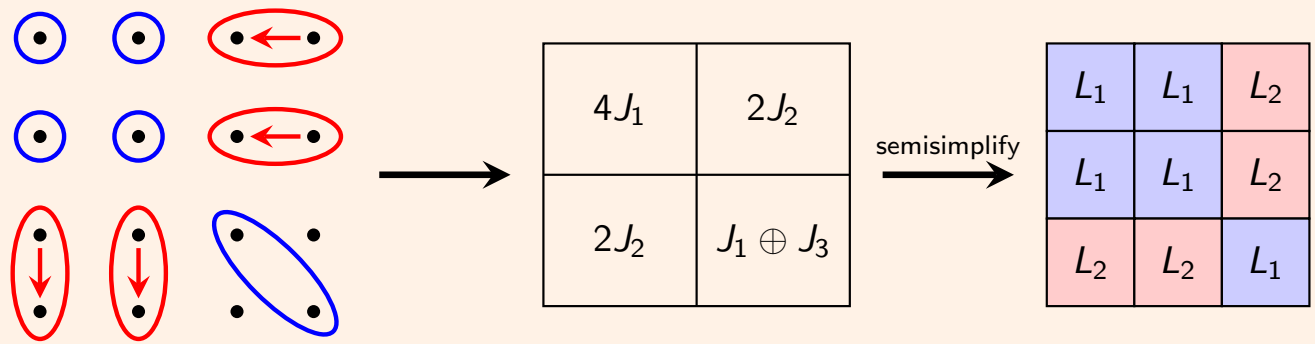
Example

The vector space $V = mJ_1 \oplus nJ_{p-1}$ is sent to the super vector space $\overline{V}$ which has super dimension $m|n$.

Semisimplification

Example ($p = 3$, Computing $\overline{\mathrm{Mat}(2J_1 \oplus J_2)}$)

- Set $t = e_{4,3}$ acting on $V = 2J_1 \oplus J_2$. The induced action on $\mathrm{Mat}(V) = V \otimes V^*$ is $t \cdot x = [e_{4,3}, x] = e_{4,3}x - xe_{4,3}$.
- For example, $e_{3,1} \to e_{4,1} \to 0$ is a J_2, $e_{3,4} \to e_{4,4} - e_{3,3} \to -2e_{4,3} \to 0$ is a J_3, and $e_{3,3} + e_{4,4} \to 0$ is a J_1.
- So, $\overline{\mathrm{Mat}(2J_1 \oplus J_2)} = \overline{5J_1 \oplus 4J_2 \oplus J_3} = 5L_1 \oplus 4L_2 \cong \mathrm{Mat}(2|1)$.



Main Results

Let $V = mJ_1 \oplus nJ_{p-1}$, so $\overline{V}$ has super dimension $m|n$.

Theorem (Polynomial functions)

As supercommutative algebras,

$$\overline{\mathcal{O}(\mathrm{Mat}(V))} \cong \mathcal{O}(\mathrm{Mat}(m|n)) \otimes k[Y_{i,j}, Z_{i,j,r}],$$

where the $2mn + (p-2)n^2$ extra generators $Y_{i,j}, Z_{i,j,r}$ are even, Frobenius twisted, and of degree p. In particular, for degree $d < p$, there are no extra generators.

- $\mathcal{O}(\mathrm{Mat}(m|n))$ contains odd generators like $x_{i,\bar{j}}$.
- By the Koszul sign rule, $x_{i,\bar{j}}^2 = -x_{i,\bar{j}}^2$, so $x_{i,\bar{j}}^2 = 0$.
- This means each odd generator can appear at most once in a monomial.

Main Results

Theorem (Coalgebra structure)

The comultiplication on $\overline{\mathcal{O}(\mathrm{Mat}(V))}$ *is inherited from* $\mathcal{O}(\mathrm{Mat}(m|n))$ *for* x_{ij}*, while* $\Delta(Y_{i,j})$ *and* $\Delta(Z_{i,j,r})$ *are sums that contain* $Y_{i,j}$ *and* $Z_{i,j,r}$ *terms, as well as the p-th powers* x_{ij}^p*.*

Theorem (Coordinate ring)

As supercommutative algebras,

$$\overline{\mathcal{O}(\mathrm{GL}(V))} \;\cong\; \mathcal{O}(\mathrm{GL}(m|n)) \otimes k[Y_{i,j}, Z_{i,j,r}].$$

Acknowledgments

I would like to express my gratitude to my mentor, Arun Kannan, for his consistent support and guidance throughout the project, for helping me develop research ideas, and for his help with my presentation. Additionally, I would like to thank Frank Wang and Gary Li for their feedback on the presentation, and Thomas Rued for his feedback on the reading report. My thanks also go to the PRIMES staff for their efforts in facilitating this program. Finally, I would like to extend my thanks to my parents for their unwavering support.

References I


David J. Benson.
Representations of Elementary Abelian p-Groups and Vector Bundles, chapter 2, pages 50–51.
Cambridge University Press, 2016.


Jonathan Brundan and Jonathan Kujawa.
A new proof of the mullineux conjecture.
Journal of Algebraic Combinatorics, 18(1):13–39, 2003.


Pavel Etingof, Shlomo Gelaki, Dmitri Nikshych, and Victor Ostrik.
Tensor Categories, volume 205 of *Mathematical Surveys and Monographs*.
American Mathematical Society, Providence, Rhode Island, 2015.


Pavel Etingof and Victor Ostrik.
On semisimplification of tensor categories.
Selecta Mathematica, 24(4):3347–3381, 2018.

References II


J. A. Green.
Polynomial Representations of GL_n, volume 830 of *Lecture Notes in Mathematics*.
Springer, 1980.

Backup: Schur Algebra Results

- The Schur algebra is $S(V,d) = A(V,d)^*$, where $A(V,d)$ is the degree d part of $\mathcal{O}(\mathrm{Mat}(V))$. Its modules are the polynomial representations of $\mathrm{GL}(n)$ of degree d.
- We calculate $\overline{S(V,d)} = \overline{A(V,d)}^*$:

Lemma (Grading)

Setting $w(x_{ij}) = 0$, $w(Y_{i,j}) = p-2$, and $w(Z_{i,j,r}) = 2(p-1-r)$, and extending additively, gives a grading of $\overline{A(V,d)}$ as a coalgebra. Dually, $\overline{S(V,d)}$ is a graded algebra.

Backup: Schur Algebra Results (cont.)

- Weights in degree d are bounded, so the positive-weight part I_d (dual to monomials containing some Y or Z) is a **nilpotent ideal**, and $\overline{S(V,d)}/I_d \cong S(m|n,d)$.

Theorem (Simple modules)

$\overline{S(V,d)}$ and $S(m|n,d)$ have the same simple modules, which are the irreducible polynomial representations of $\mathrm{GL}(m|n)$ of degree d.

- I_d is nilpotent, so it acts by zero on any simple module M, as $I_d M$ is a submodule and $I_d M = M$ would give $M = I_d^k M = 0$.