On Signature Characters and Unitary Loci for Type A_1, A_2 Quantum Groups at Roots of Unity

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Lie Algebras and Representations

Lie Algebras

A **Lie algebra** $\mathfrak{g}$ over a field k is a vector space with a **Lie bracket**, or a bilinear map $[\cdot,\cdot]:\mathfrak{g}\times\mathfrak{g}\to\mathfrak{g}$, satisfying:

- $[x,y]=-[y,x]$ (skew-symmetry)
- $[x,[y,z]]+[y,[z,x]]+[z,[x,y]]=0$ (Jacobi identity)

Example ($\mathfrak{sl}_2(k)$)

All 2×2 matrices with trace 0, or a vector space with basis

$$H=\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix},\quad E=\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix},\quad F=\begin{bmatrix}0 & 0\\ 1 & 0\end{bmatrix}.$$

Example ($\mathfrak{gl}_n(k)$)

All $n\times n$ matrices with k-valued entries.

Both have a Lie bracket $[x,y]=xy-yx$.

Representations of Lie Algebras

Let $\mathfrak{g}$ be a Lie algebra and V be a vector space. A **representation** of $\mathfrak{g}$, denoted as (ρ, V), is a Lie algebra homomorphism $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$.

Example (Representation of $\mathfrak{sl}_2(\mathbb{C})$)

Let $\{v_0, \ldots, v_m\}$ denote the basis of V.

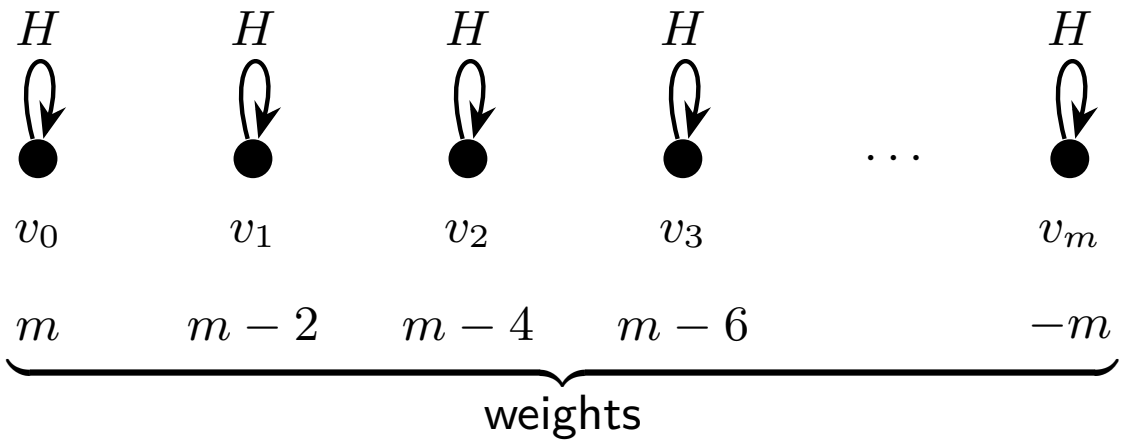


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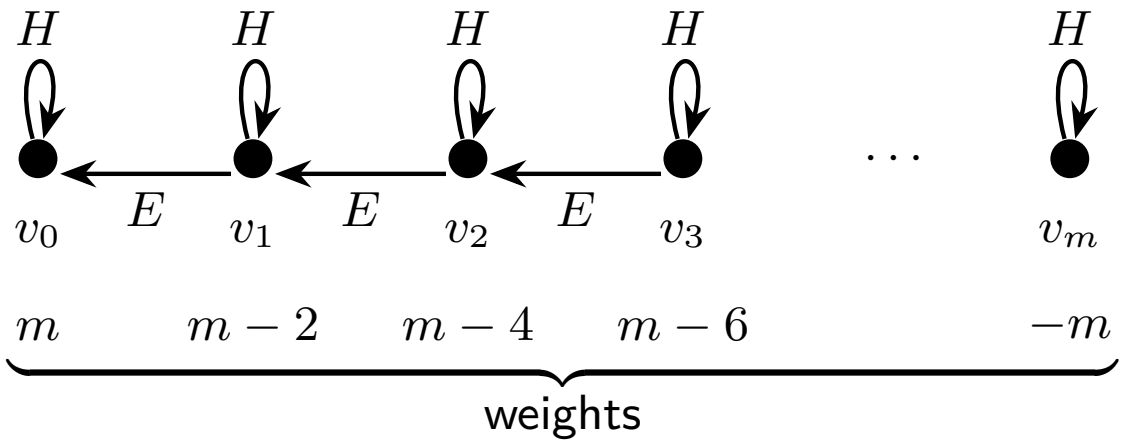
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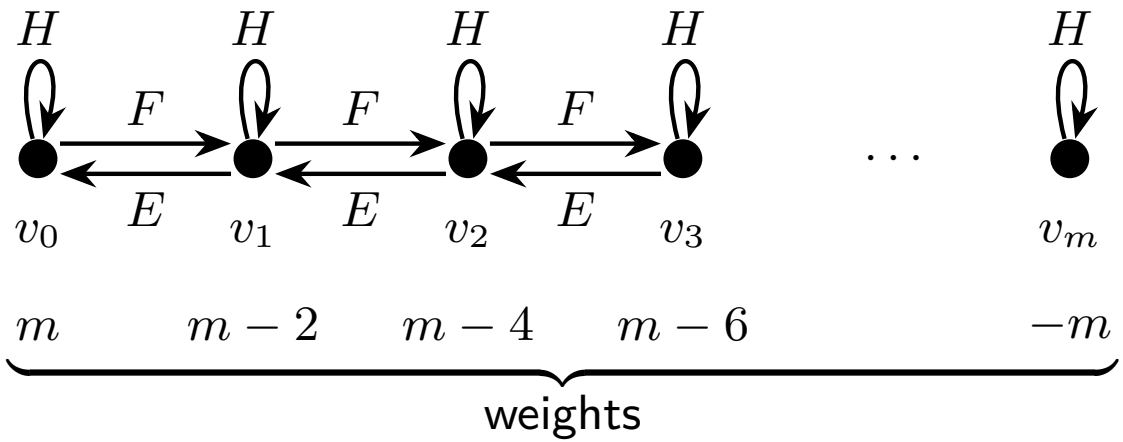
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- $HE - EH = 2E$

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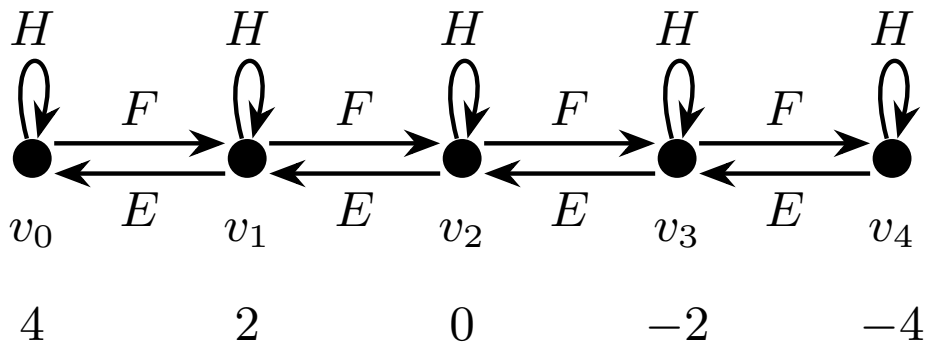
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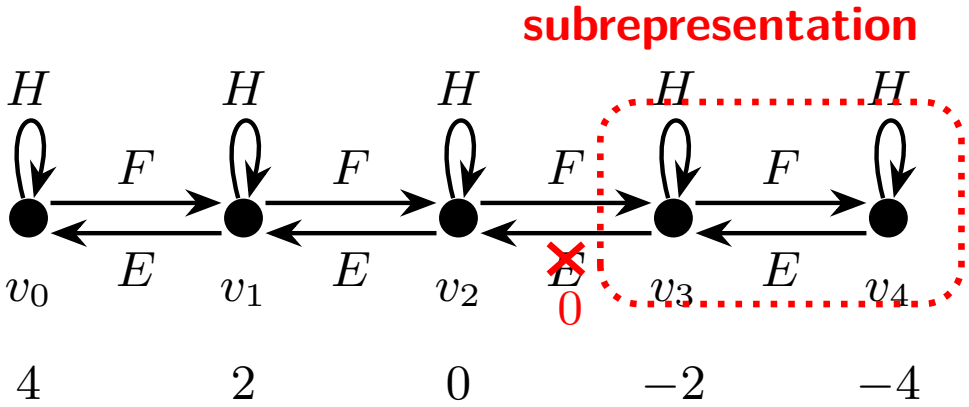
- $Hv_i = (m - 2i)v_i$
- $HE - EH = 2E$
- $HF - FH = -2F, \quad EF - FE = H$

What Happens in Characteristic p?

Example: $m = 4$, over Char 0



Example: $m = 4$, over Char 3



Representations of Quantum Groups

Representations of $U(\mathfrak{sl}_2)$ and $U_q(\mathfrak{sl}_2)$

Given the Lie algebra $\mathfrak{sl}_2$:

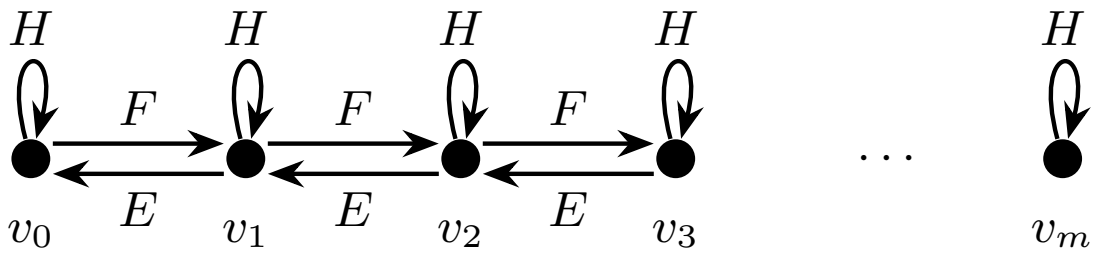
- $U(\mathfrak{sl}_2)$ is its **universal enveloping algebra**.
  - Generators: E, F, H.
  - Relations: $HE - EH = 2E, \quad HF - FH = -2F, \quad EF - FE = H$
- $U_q(\mathfrak{sl}_2)$ is its **quantized enveloping algebra** ($q \in \mathbb{C}$).
  - Generators: $E, F, K^{\pm 1}$, where $K = q^H$.
  - Relations: $KK^{-1} = K^{-1}K = 1$,
    $KEK^{-1} = q^2E, \quad KFK^{-1} = q^{-2}F, \quad EF - FE = \frac{K - K^{-1}}{q - q^{-1}}$.

Moving from $U(\mathfrak{sl}_2)$ to $U_q(\mathfrak{sl}_2)$:

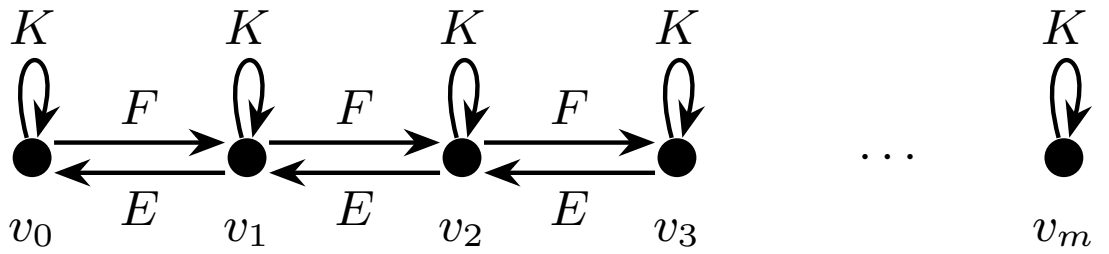
- Highest-weight m becomes q^m.
- Actions of E, F become their corresponding q-**integers**.
  - $[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}}$ where $n \in \mathbb{Z}$.

Representations of $U(\mathfrak{sl}_2)$ and $U_q(\mathfrak{sl}_2)$

1. Weight diagram for the irreducible representations of $U(\mathfrak{sl}_2)$:

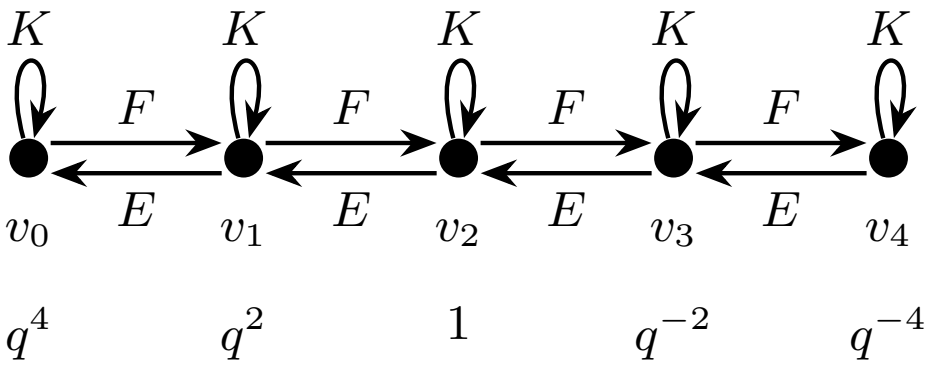


2. Weight diagram for the irreducible representations of $U_q(\mathfrak{sl}_2)$:

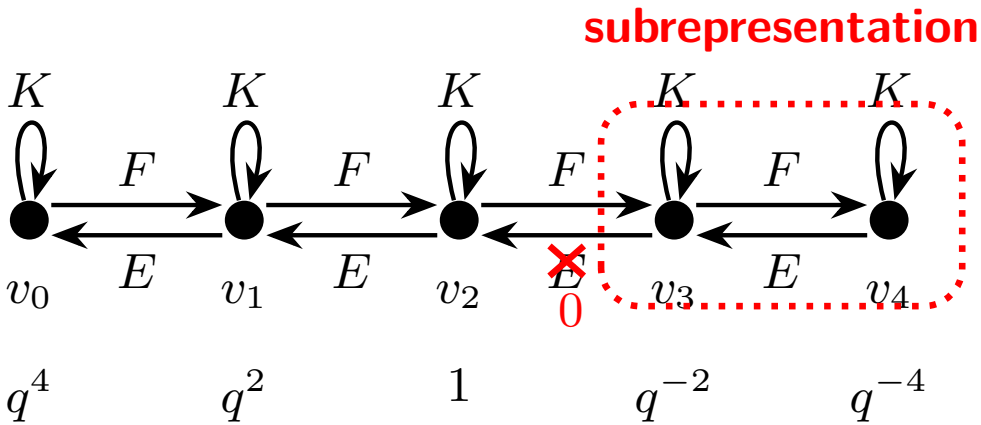


What Happens at Roots of Unity?

Example: $m = 4$, generic q



Example: $m = 4$, $q^3 = 1$, $q \neq 1$ (third root of unity)



Three Types of Quantum Groups

Actually, at roots of unity, there are three types of $U_q(\mathfrak{sl}_2)$!

De Concini–Kac Quantum Group

Directly specialize q to a root of unity (what we just looked at is one case of it)

Small Quantum Group

A finite-dimensional quotient of the De Concini–Kac quantum group.

Lusztig Quantum Group

Keeps additional Lusztig's "divided-power" operators.

Unitary Representations of Quantum Groups

Hermitian Forms

Definition:

- Let V be a $U_q(\mathfrak{sl}_2)$-representation. A **Hermitian form** on a vector space V over the field $\mathbb{C}$ is a map $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{C}$ such that for all $u, v, w \in V$ and all $a, b \in \mathbb{C}$:
  - $\langle u, v \rangle = \overline{\langle v, u \rangle}$.
  - $\langle au + bv, w \rangle = a\langle u, w \rangle + b\langle v, w \rangle$.
  - $\langle u, av + bw \rangle = \overline{a}\langle u, v \rangle + \overline{b}\langle u, w \rangle$.

Properties:

- The form is **positive-definite** if $\langle v, v \rangle > 0$ for all $v \neq 0$
- The form is **invariant** if $\langle Xu, v \rangle = \langle u, X^* v \rangle$ for all $X \in U_q(\mathfrak{sl}_2)$ and $u, v \in V$, where $E^* = F, \quad F^* = E, \quad K^* = K^{-1}, \quad q^* = q^{-1}$.
  - It requires $|q| = 1$ and $|\lambda| = 1$ for every weight λ.

Unitary Representations

- For each weight space of a $U_q(\mathfrak{sl}_2)$-representation, we apply an invariant Hermitian form $\langle \cdot, \cdot \rangle$ and see if it's positive or negative.
- We use a tool called **signature character** to record these signs across all weight spaces.
- The representation is **unitary** if $\langle \cdot, \cdot \rangle$ is positive-definite for all weight spaces.
- **Unitary locus** is the range of parameters (e.g. highest-weight) admitting a unitary representation.

Problem and Results

Previous Work

Venkateswaran (2014) studied signature characters for highest-weight representations of Type-A_{n-1} quantum group $U_q(\mathfrak{sl}_n)$ when q is generic.

- Derived explicit signature characters and unitary loci.
- Constructed a Gelfand–Tsetlin basis (orthogonal).

Since representations of $U_q(\mathfrak{sl}_n)$ drastically differ at roots of unity, this naturally motivates the question:

Can we compute unitary loci when q is a root of unity?

Results Outline

YES! We derived closed-form signature characters and unitary loci for the following representations in the root-of-unity setting:

Type A_1 Quantum Group ($U_q(\mathfrak{sl}_2)$):

1. De Concini-Kac Quantum Group
   - Highest-weight representations with dimensions $< \mathrm{ord}(q^2)$.
   - Fully-cyclic representations with dimensions $\mathrm{ord}(q^2)$.
   - Highest- and lowest-weight representations with dimensions $\mathrm{ord}(q^2)$.
2. Small Quantum Group
3. Lusztig Quantum Group

Type A_2 Quantum Group ($U_q(\mathfrak{sl}_3)$):

1. Lusztig Quantum Group: representations with highest-weights in a restricted range

Selected Results

Theorem (2026, Li)

Fix $k \in \mathbb{Z}_{>0}$ and $\ell = \mathrm{ord}(q^2)$. Let $L(k)$ denote the Lusztig $U_q(\mathfrak{sl}_2)$-representation with highest-weight k. The set of roots of unity $q = e^{2\pi i p/n} \neq \pm 1$ with $\gcd(p,n) = 1$ for which $L(k)$ is unitary is given by:

1. *$\ell > k$, and $\left|\frac{p}{n}\right|_{\mathbb{R}/\mathbb{Z}} < \frac{1}{2k}$.*
2. *$\ell \leq k$, and satisfies either $\ell \mid k$ or $\left|\frac{p}{n}\right|_{\mathbb{R}/\mathbb{Z}} < \frac{1}{2(k \mod \ell)}$.*

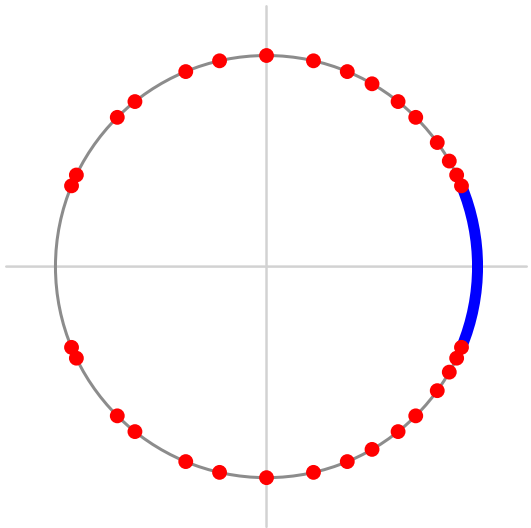


Figure: Roots of unity admitting a unitary $L_q(8)$.

Future Work

Future work includes deriving closed-form signature character formulas and unitary loci for the following at roots of unity:

1. All weights for Lusztig $U_q(\mathfrak{sl}_3)$ representations.
2. De Concini-Kac quantum groups $U_q^{DK}(\mathfrak{sl}_n)$ for $n \geq 3$ through a newly introduced Gelfand-Tsetlin basis.
3. Ultimately, **all** Type-A quantum groups (in Lusztig's form).
4. Potentially even Type-B and Type-C quantum groups.

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THANK YOU FOR LISTENING!

Questions?

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[1] Vidya Venkateswaran. Signature characters of highest-weight representations of $U_q(\mathfrak{gl}_n)$. *Algebras and Representation Theory*, 19(2):473–487, 2016.