# Probabilistic Fault-Tolerant Metric Dimension

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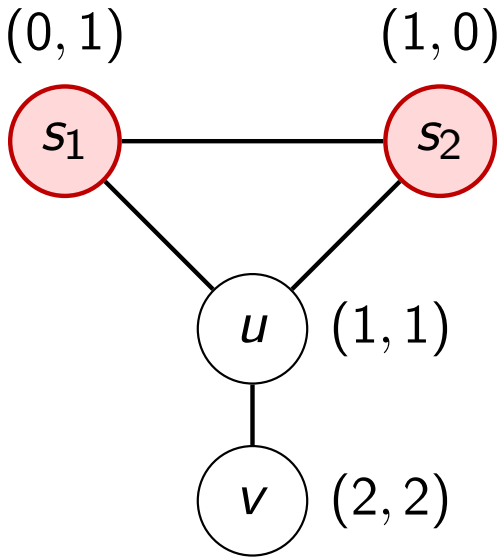
# Identifying vertices using distances

Choose a set of landmarks $S = \{s_1, \ldots, s_k\}$ in a graph $G$. Each vertex $v$ has a distance vector

$$d_S(v) = (d(v, s_1), \ldots, d(v, s_k)).$$

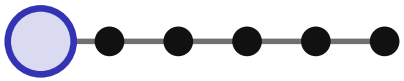
- $S$ is a **resolving set** if every vertex has a different vector.
- The **metric dimension** $\dim(G)$ is the smallest possible $|S|$.

**Example: a triangle with a leaf.** Take $S = \{s_1, s_2\}$; labels show $d_S(v)$.



# Metric dimension for elementary graph classes

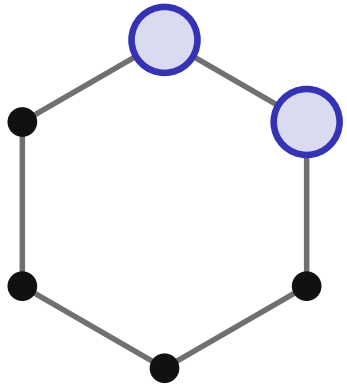
**Paths**



$\dim(P_n) = 1$

One dist. coordinate.

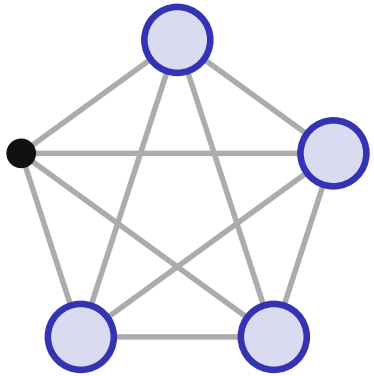
**Cycles**



$\dim(C_n) = 2$

Break the symmetry.

**Complete graphs**



$\dim(K_n) = n - 1$

Nearly every vertex.

Blue vertices form a minimum resolving set in each example.

$n \geq 2$ for paths and complete graphs; $n \geq 3$ for cycles.

**Where do resolving sets appear?**

Graph localization · Chemical graphs · Error-correcting codes · Static Mastermind

# A mature theory: structure, probability, computation

Established since the 1970s; benchmarks across several kinds of mathematics.

**Order and diameter**

$n \leq D^{\dim(G)} + \dim(G)$

Few coordinates limit graph size.

**Random graphs**

$G \sim G(n, \frac{1}{2})$:
$\dim(G) \sim 2 \log_2 n$
with high probability.

**Computation**

Decide whether $\dim(G) \leq k$:
**NP-complete**.

**Robustness and variants**

$\dim^{(2)}(G) = \text{ftdim}(G)$

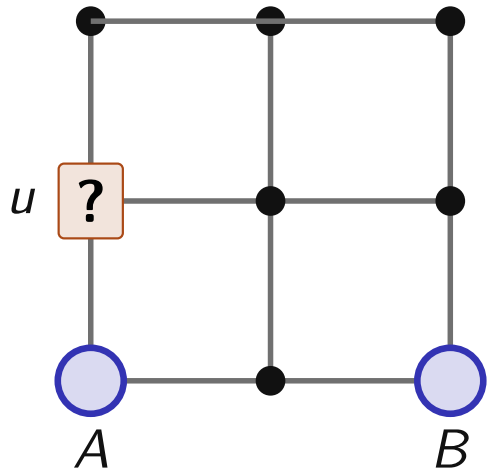
Tolerate any one deletion.

Edge, mixed, fault-tolerant.

$n = |V(G)| \geq 2$; $D$ is diameter. $\dim^{(k)}$: $k$ distinguishers per pair.
*With high probability*: probability tends to one as $n \to \infty$.

# A missing sensor can erase a landmark

Distance reports can locate a robot. Sensors can fail.




Blue circles: installed sensors.

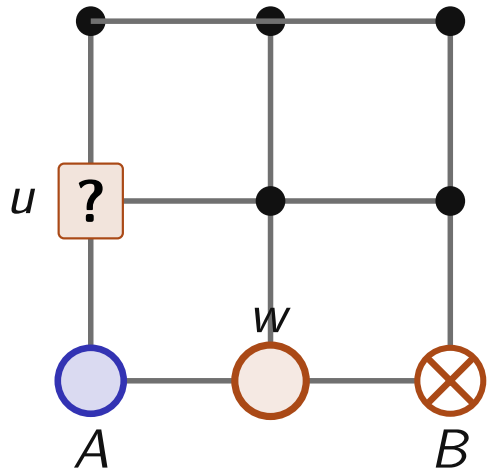
**Both sensors report.**

$$d_S(u) = (1, 3).$$

Every location has a different pair of readings.

# A missing sensor can erase a landmark

Distance reports can locate a robot. Sensors can fail.




Blue circles: installed sensors.

**Sensor $B$ is silent.**

$$d(u, A) = d(w, A) = 1.$$

Two locations now give the same available report.

## From identification to reliability

If landmarks fail independently, how few can keep the **whole graph** resolvable with a prescribed confidence?

# Probabilistic fault-tolerant metric dimension

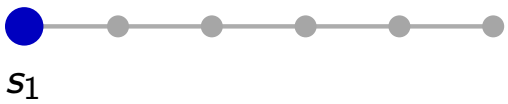
Each selected landmark fails **independently** with probability $q$. Let $S'$ be the surviving subset of $S$.

$$\rho_q(S) = \mathbb{P}\big(S' \text{ resolves } G\big), \qquad \dim_p(G, q) = \min\big\{|S| : \rho_q(S) \geq p\big\}.$$

- $q$ measures landmark unreliability. $p$ is the required confidence.
- Success means distinguishing **every pair of vertices at once**.
- If even selecting all vertices cannot reach $p$, the value is $+\infty$.

**Path example:** $q = 0.1$. Blue vertices are installed landmarks.

**One endpoint**




$$\rho_q(\{s_1\}) = 1 - q = \boxed{0.9}.$$

The endpoint must survive.

**Both endpoints**




$$\rho_q(\{s_1, s_2\}) = 1 - q^2 = \boxed{0.99}.$$

At least one endpoint must survive.

# Reliability comes from resolving subsets

Fix a deployment $S$ with $|S| = k$. Let $r_t(S)$ count its resolving subsets containing exactly $t$ landmarks.

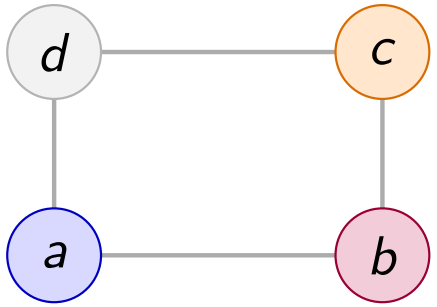
$$\rho_q(S) = \sum_{t=0}^{k} r_t(S)(1-q)^t q^{k-t}$$

- A survivor set works exactly when it contains an inclusion-minimal resolving set.
- How these minimal sets **overlap** matters: their success events can overlap.

**For the pictured example:**

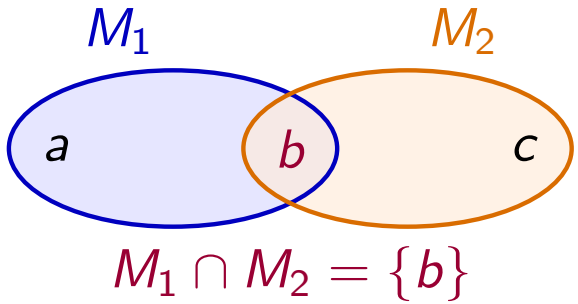
$$\rho_q(S) = 2(1-q)^2 q + (1-q)^3.$$

**Example:** $C_4$, $S = \{a, b, c\}$



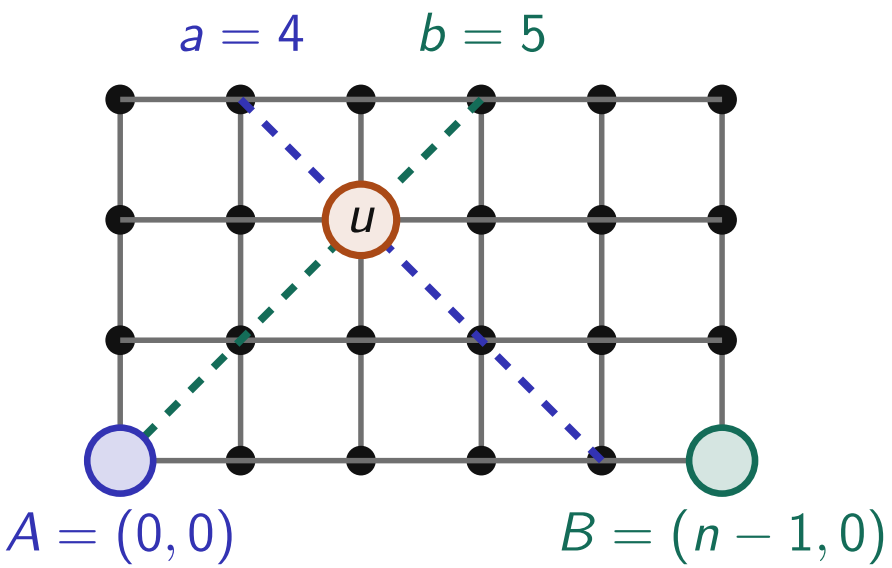

$r_2(S) = 2, \qquad r_3(S) = 1.$

Minimal resolving subsets:



# Classical grids: two distances determine two coordinates

For $m, n \geq 2$, $\boxed{\dim(P_m \square P_n) = 2}$.




Example: $P_4 \square P_6$.

Dashed contours meet at one vertex.
One landmark cannot suffice: the grid is not a path.

For $u = (x, y)$, its readings are

$$a = x + y, \qquad b = n - 1 - x + y.$$

Sum determines the row:

$$a + b = n - 1 + 2y.$$

Difference determines the column:

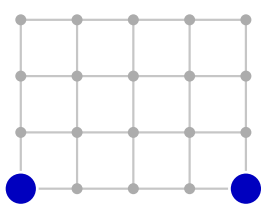
$$a - b = 2x - (n - 1).$$

**From identification to reliability**

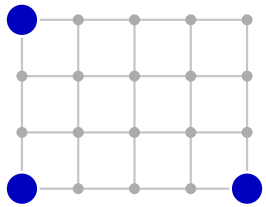
Two landmarks resolve the grid. What if either disappears?

# Grids: the first optimal deployments

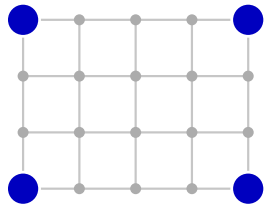
An $m \times n$ grid has distance equal to horizontal plus vertical steps. For $m, n \geq 3$, its metric dimension is 2. The resolving pairs are exactly the pairs of corners joined by a side.



$k = 2$



$k = 3$



$k = 4$

| $k$ | Optimal placement | Maximum success probability | $q = 0.1$ |
|---|---|---|---|
| 2 | Two corners on one side | $(1-q)^2$ | 81% |
| 3 | Any three corners | $(1-q)^2(1+q)$ | 89.1% |
| 4 | All four corners | $(1-q)^2(1+q)^2$ | 98.01% |

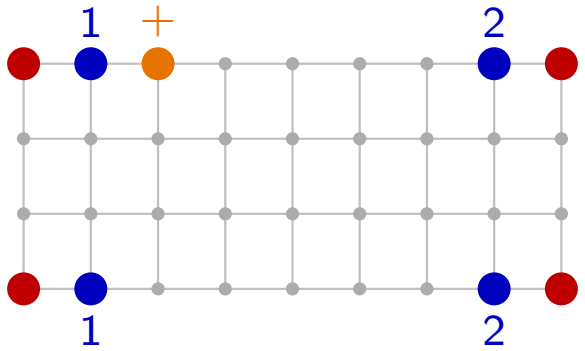
These optima are independent of the grid dimensions.

At $q = 0.1$, a 95% confidence requirement needs exactly **four landmarks**.

# Larger grid deployments: a conjectured optimal family

Assume $3 \leq m \leq n$ and $4 \leq k \leq 2n$.

- Begin with all four corners.
- Add matched vertices on the top and bottom long sides, following the canonical order from the ends inward.
- For odd $k$, add one extra long-side vertex.




Example: $k = 9$

Corners Matched pairs Extra

**Conjecture:** the canonical set $C_k$ maximizes the number of resolving subsets at every size:

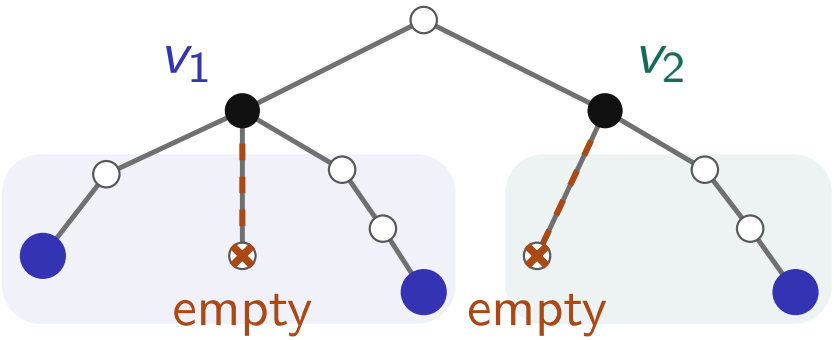
$$r_t(S) \leq r_t(C_k) \qquad \text{for every } k\text{-set } S \text{ and every } t.$$

**Established:** exact counting and reliability formulas for these families.

**Evidence:** reported exhaustive checks on $3\times3$, $3\times4$, $3\times5$, $4\times4$, and $4\times5$ grids.

# Terminal branches turn success into a product

Non-path tree $T$




• surviving landmark $\times$ failed

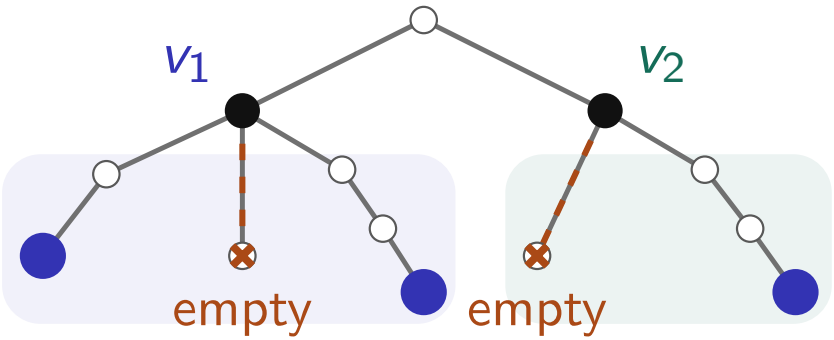
Terminal branch: leaf to its nearest major vertex, excluding that vertex.

**Classical criterion**

Resolves iff at least $t-1$ of the $t$ branches in each group are met.

# Terminal branches turn success into a product

Non-path tree $T$; independent failure probability $q \in (0,1)$.




• surviving landmark × failed

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## After failures

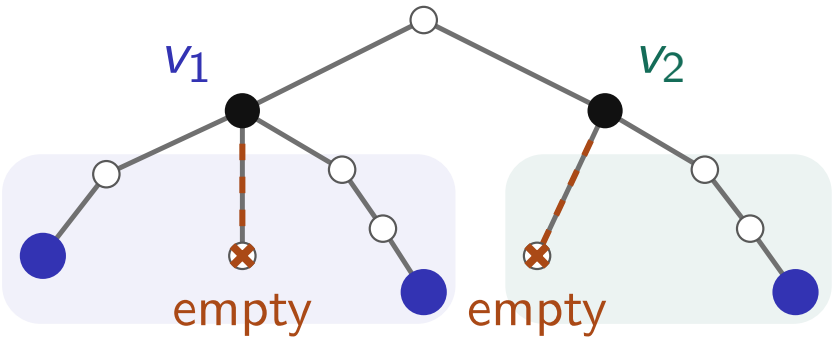
At most **one empty branch** per group.

$m_j$: landmarks selected on branch $j$.

$$R_q(m_1,\dots,m_t) = \underbrace{\prod_j (1-q^{m_j})}_{\text{no branch empty}} + \underbrace{\sum_j q^{m_j} \prod_{\ell \neq j} (1-q^{m_\ell})}_{\text{exactly one branch empty}}.$$

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Disjoint branch groups $\Longrightarrow$ independent local events

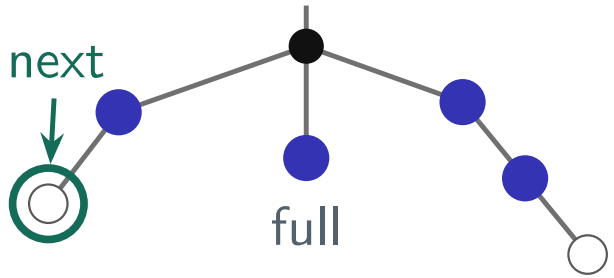
$\boxed{\rho_q(S) = \prod_i R_q(\mathbf{m}_i)}$ $\mathbf{m}_i$: counts in group $i$

**How should we allocate landmarks to maximize this product?**

# Balance locally; take the largest available gain

Fix $q$. Start with one landmark on $t_i - 1$ branches in each group.

## 1. Balance locally



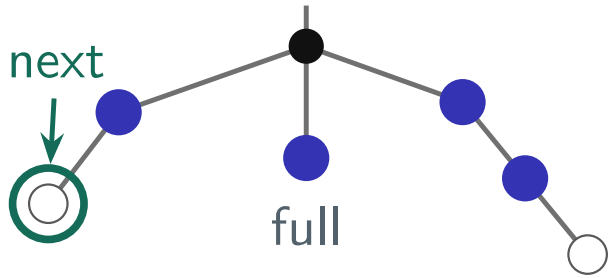

$(1, 1, 2) \to (2, 1, 2)$

Add to a least-filled branch with room.

# Balance locally; take the largest available gain

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## 1. Balance locally




$(1, 1, 2) \to (2, 1, 2)$

Add to a least-filled branch with room.

## 2. Gains monotonically decrease

$P_i(c)$: best success in group $i$ with $c$ landmarks.

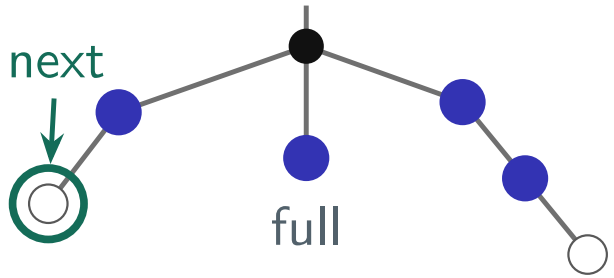
$$\gamma_{i,c} = \frac{P_i(c+1)}{P_i(c)}$$

$\gamma_{i,c} \geq \gamma_{i,c+1}$

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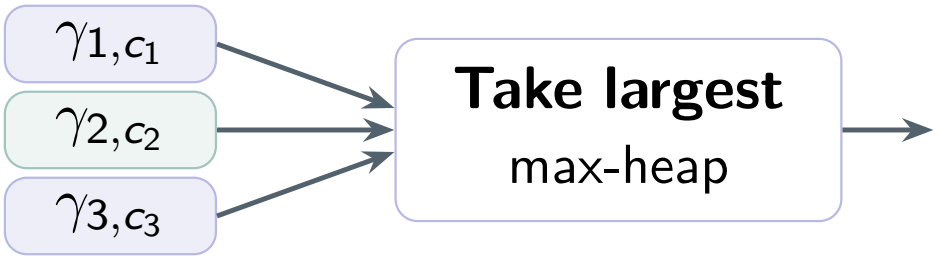
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$\gamma_{i,c} \geq \gamma_{i,c+1}$

## 3. Choose the largest gain globally




Add one landmark
Update its gain; repeat.

**Nested optimal deployments at every size**

$O(N \log N)$ arithmetic operations and comparisons

Exact polynomial bit complexity for rational $p, q$; $N = |V(T)|$.

# A lower bound from surviving landmarks

**Observation.** If the surviving landmarks resolve $G$, there must be at least $\dim(G)$ of them.

$$\{\text{survivors resolve } G\} \subseteq \{|S'| \geq \dim(G)\}.$$

For a landmark set $S$ of size $m$, independent failures give

$$|S'| \sim \mathrm{Bin}(m, 1-q), \qquad \rho_q(S) \leq \mathbb{P}(\mathrm{Bin}(m, 1-q) \geq \dim(G)).$$

To achieve any fixed success probability, the binomial tail forces

$$m \geq \frac{\dim(G)}{1-q} - O\left(\sqrt{\dim(G)}\right).$$

This ignores where the survivors land, so the bound applies to every deployment.

# An upper bound by averaging over landmark sets

**Idea.** Choose $S$ uniformly among all $m$-vertex landmark sets, then allow its landmarks to fail.

**Symmetry:** Conditional on $t$ survivors, the surviving $t$-set is uniform.

| Selected landmarks | Survivors |
|---|---|
| $\{a, b, c\}$ | $\{a, b\}$ |
| $\{a, b, d\}$ | $\{a, b\}$ |
| $\{a, b, e\}$ | $\{a, b\}$ |

If $\{a, b\}$ survive from three selected landmarks, the failed one could be $c$, $d$, or $e$. There are equally many choices for *any* surviving pair.

Let $R_t$ be a uniformly random $t$-set. Since adding landmarks preserves resolution,

$$\mathbb{E}_{|S|=m}[\rho_q(S)] \;\geq\; \mathbb{P}(\mathrm{Bin}(m, 1-q) \geq t)\mathbb{P}(R_t \text{ resolves } G).$$

**Existence:** Some fixed landmark set does at least as well as the average.

# When do the bounds match?

The bounds agree if random landmark sets near the classical optimum already resolve the graph. Suppose that, for a growing graph family,

$$\mathbb{P}(R_t \text{ resolves } G) \longrightarrow 1 \qquad \text{at } t = (1 + o(1))\dim(G).$$

To retain $t$ landmarks with fixed confidence, we need to select roughly

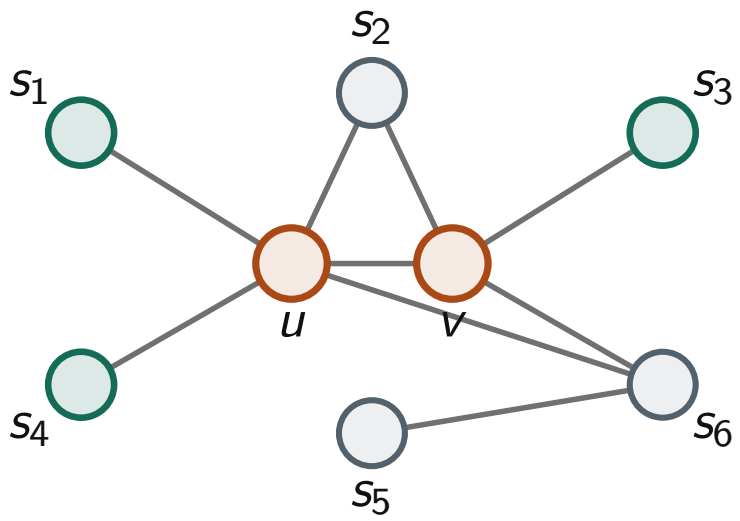
$$\frac{t}{1-q} + O(\sqrt{t}).$$

Combining the two bounds, when $\dim(G) \to \infty$ and $\dim(G) = o(n)$,

$$\dim_p(G, q) = \left(\frac{1}{1-q} + o(1)\right)\dim(G).$$

# Distinguishing pairs in dense random graphs

In $G(n, \pi)$, each possible edge is present independently with probability $\pi$.




Green: one edge to $u, v$ Gray: zero or two

An outside landmark distinguishes $u, v$ if it is adjacent to exactly one of them. The chance it *fails* this test is

$$\theta = \pi^2 + (1 - \pi)^2.$$

Different adjacency also means different distances.

**Known result without failures** (with high probability):

$$\dim(G) = \left( \frac{2}{\log(1/\theta)} + o(1) \right) \log n.$$

# Why random landmarks are nearly optimal

For a pair $u, v$, each outside landmark gives an adjacency bit (1 for an edge, 0 otherwise). The pair is missed only if all the bits agree.

| | $s_1$ | $s_2$ | $s_3$ | | $s_t$ |
|---|---|---|---|---|---|
| $u$ | 1 | 0 | 1 | $\dots$ | 0 |
| $v$ | 1 | 0 | 1 | | 0 |

Each bit matches with probability $\theta$, independently across landmarks:

$$\mathbb{P}(\text{a fixed pair is missed}) \leq \theta^t.$$

Union bound over all vertex pairs:

$$\mathbb{P}(\text{some pair is missed}) \leq \binom{n}{2}\theta^t.$$

At $t = \frac{2\log n}{\log(1/\theta)} + \sqrt{\log n}$, the upper bound tends to zero.

# Metric dimension of dense random graphs with failures

**Our result.** For fixed $\pi, p, q \in (0,1)$, with high probability for $G \sim G(n,\pi)$,

$$\dim_p(G,q) = \left(\frac{2}{(1-q)\log(1/\theta)} + o(1)\right)\log n.$$

**Why this coefficient is sharp:**

- The lower bound requires roughly $\dim(G)/(1-q)$ selected landmarks.
- The random-set argument shows that this many are sufficient to leading order.

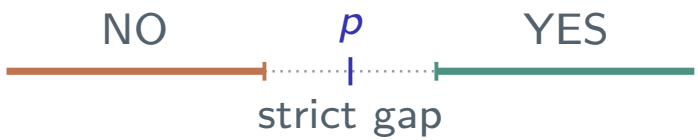
Equivalently,

$$\frac{\dim_p(G,q)}{\dim(G)} \longrightarrow \frac{1}{1-q}.$$

# Failures create a counting problem

Fix a deployment $S$. At $q = \frac{1}{2}$, every $U \subseteq S$ is equally likely.

$$\rho_{1/2}(S) = \frac{\#\{U \subseteq S : U \text{ resolves } G\}}{2^{|S|}}.$$




Compute $\rho_q(S)$ exactly

**EVAL-PMD**

**#P-hard**

#SAT analogy:
count successful witnesses

Decide whether $\rho_q(S) \geq p$

**FEAS-PMD**

**PP-complete**

MAJSAT analogy:
compare the count to a threshold

**Both hardness results already hold at $q = \frac{1}{2}$.**

# Choosing the deployment adds an existential layer

$$\dim_p(G, q) \leq k \ ?$$

**PMD-Decision is $\mathbf{NP^{PP}}$-complete.**

**Choose $S$**

$|S| \leq k$



**Threshold over failures**

Does $\rho_q(S)$ reach $p$?

E-MAJSAT analogy

$$\exists x : \quad \#\{y : \Phi(x, y) = 1\} \geq T.$$

$x$: deployment choice $y$: encoded survival configuration

**Design choice and counting uncertainty: both layers matter.**

Fix either rational $p$ or $q$ in $(0, 1)$: still complete.
The other remains input; both fixed is open in general.

# Sample causes of failure

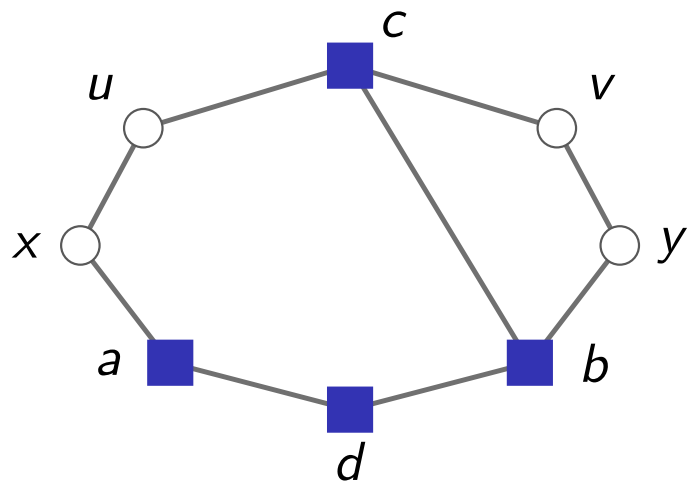
Fix $S$ in an $n$-vertex graph; estimate $F = 1 - \rho_q(S)$.

$2^{|S|}$ possible surviving sets $\quad K \leq \binom{n}{2}$ failure causes

$w_i$: probability all landmarks distinguishing pair $i$ fail; $\quad W = \sum_i w_i$.

**One randomized trial**

**Initially: $S = \{a, b, c, d\}$ active**




■ landmark dashed: unresolved

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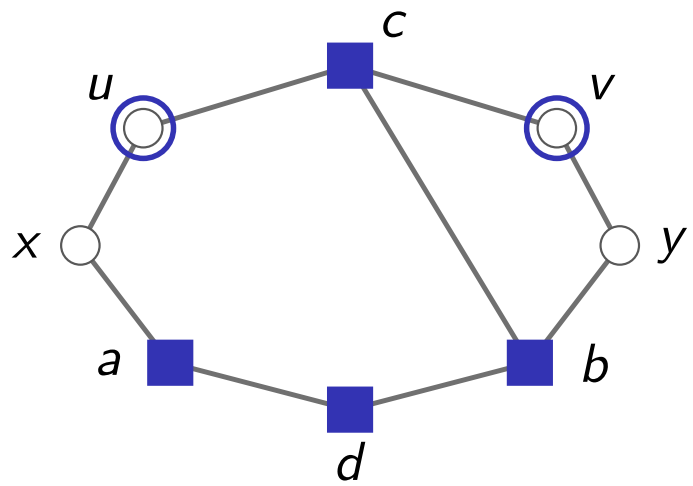
**One randomized trial**

**1. Choose a vertex pair**

Choose $i$ with probability $w_i/W$.

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chosen cause $\{u, v\}$




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## One randomized trial

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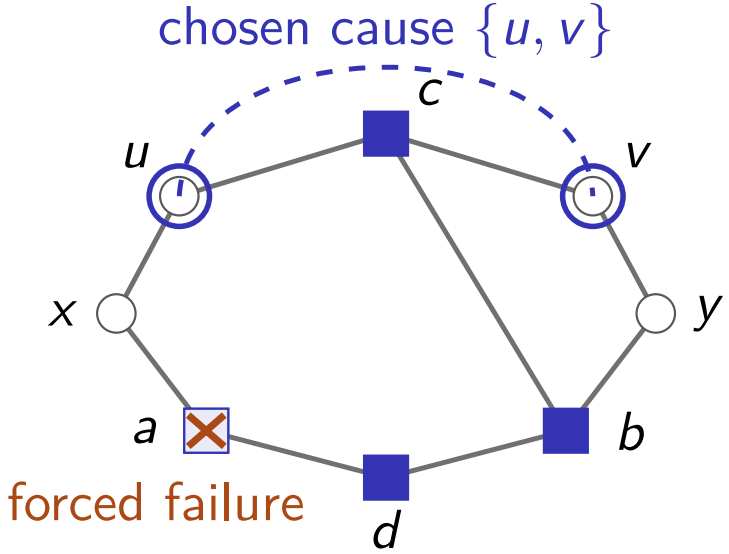
Choose $i$ with probability $w_i/W$.

**2. Force the chosen cause**

Fail its distinguishing landmarks;

$A_{uv} = \{a\}$

**Initially: $S = \{a, b, c, d\}$ active**




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# Sample causes of failure

Fix $S$ in an $n$-vertex graph; estimate $F = 1 - \rho_q(S)$.

$2^{|S|}$ possible surviving sets $\quad K \le \binom{n}{2}$ failure causes

$w_i$: probability all landmarks distinguishing pair $i$ fail; $\quad W = \sum_i w_i$.

## One randomized trial

### 1. Choose a vertex pair

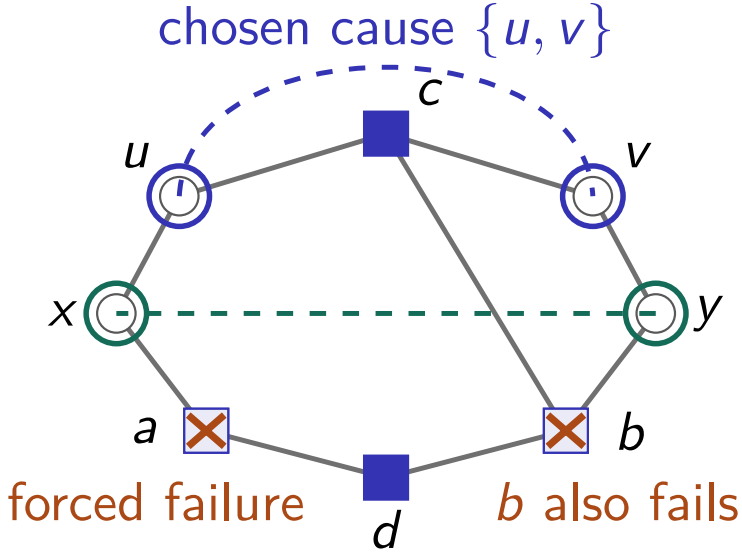
Choose $i$ with probability $w_i/W$.

### 2. Force the chosen cause

Fail its distinguishing landmarks; sample the others normally.

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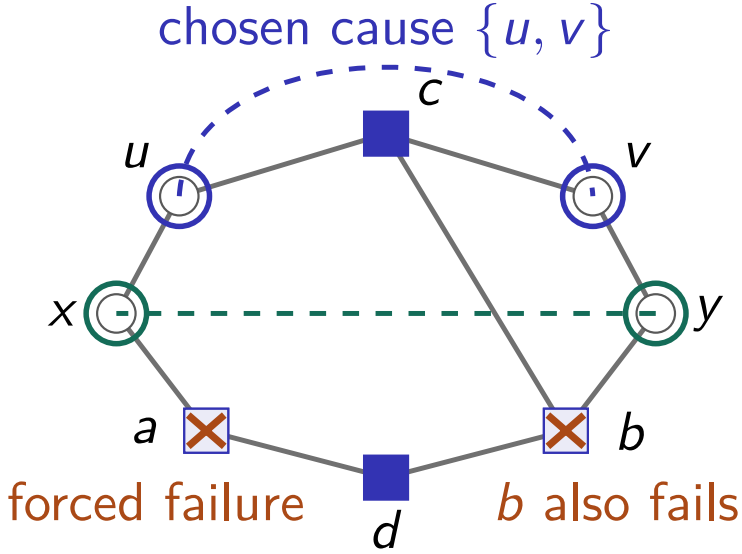
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Output $\boxed{Z = W/M}$.

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$\boldsymbol{M = 2} \implies Z = W/2$

Two cause-first routes to this config.

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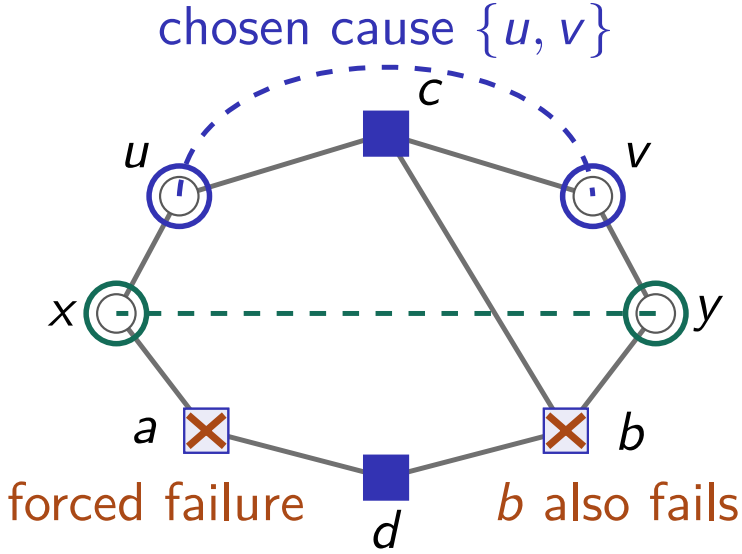
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**Average independent trial outputs to estimate $F$.**

# Why rare failures remain efficiently approximable

**Cause-first bound: independent of failure rarity.**

## Naive Monte Carlo

$$I = \mathbf{1}_{\{\text{failure}\}}$$

$$\frac{\mathrm{Var}(I)}{F^2} = \frac{1-F}{F}$$

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**Failure probability $F$ has an FPRAS.**

$$\Pr\left(|\widehat{F} - F| \le \varepsilon F\right) \ge 1-\delta$$

Time polynomial in input size, $\varepsilon^{-1}$ and $\log(\delta^{-1})$.

# Acknowledgements

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