Atomic Density and Local Behavior of Irreducible Polynomials

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Motivation/Outline

We study irreducibility in $\mathbb{N}_0[x]$, the set of polynomials with nonnegative integer coefficients. Unlike polynomials over a field, this semiring is not closed under subtraction, leading to different factorization behavior.

We study irreducibility from three connected viewpoints:

1. Global abundance: How common are irreducibles in $\mathbb{N}_0[x]$?
2. Additive representations: How often can a polynomial be written as a sum of two irreducibles?
3. Local spacing: How much must a polynomial change to become irreducible?

We begin by studying how common irreducibles are, then investigate their additive representations, and finally turn to how closely they are spaced.

Background: $\mathbb{N}_0[x]$ and Irreducibility

The nonnegative integers are denoted by $\mathbb{N}_0 = \{0, 1, 2, \ldots\}$.

- The polynomial extension $\mathbb{N}_0[x]$ consists of polynomials with nonnegative integer coefficients.
- We also write $\mathbb{Q}_{\geq 0}[x]$ for polynomials with nonnegative rational coefficients.
- A polynomial $f \in \mathbb{N}_0[x]$, other than 0 or 1, is irreducible if every factorization $f = gh$ with $g, h \in \mathbb{N}_0[x]$ has a factor equal to 1. It is reducible otherwise.

Examples

- $x^3 + 3x + 2 \in \mathbb{N}_0[x]$, while $-x + 1 \notin \mathbb{N}_0[x]$.
- $x^2 + 7x + 12 = (x + 3)(x + 4)$ is reducible in $\mathbb{N}_0[x]$.
- $x^3 + 1 = (x + 1)(x^2 - x + 1)$ is reducible in $\mathbb{Z}[x]$, but is irreducible in $\mathbb{N}_0[x]$.

Measuring Atomic Density

For a degree bound $d \geq 1$ and a coefficient bound $N \geq 1$:

- $\mathcal{T}(d, N)$ counts all polynomials in $\mathbb{N}_0[x]$ with degree at most d and coefficients at most N, including the zero polynomial.
- $\mathcal{I}(d, N)$ counts the irreducibles among these polynomials.

The proportion of irreducibles in this box is

$$\alpha(d, N) = \frac{\mathcal{I}(d, N)}{\mathcal{T}(d, N)}.$$

The atomic density is described by what happens to $\alpha(d, N)$ as the degree and coefficient bounds grow.

Example

When $d = N = 1$, the four polynomials are 0, 1, x, and $x + 1$. The last two are irreducible, so $\alpha(1, 1) = 1/2$.

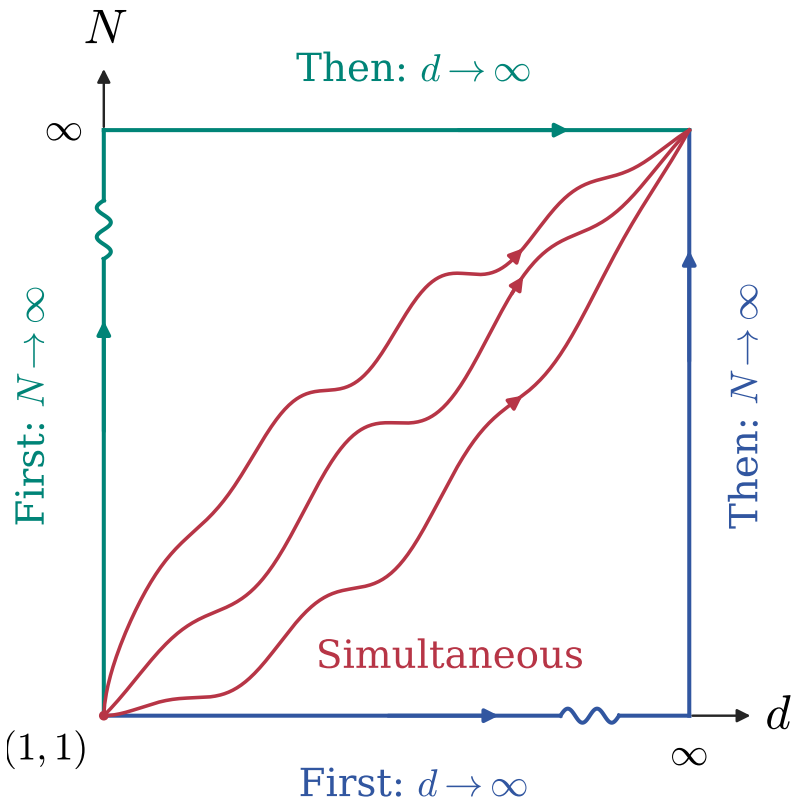
Theorem (Kolekar, Qiu, Wang)

$$\lim_{d\to\infty}\lim_{N\to\infty}\alpha(d,N)=\lim_{N\to\infty}\lim_{d\to\infty}\alpha(d,N)=1.$$

Theorem (Mittal, Pieleanu, Yan)

$$\lim_{\substack{d\to\infty\\N\to\infty}}\alpha(d,N)=1.$$

The degree and coefficient bounds can grow at any relative rates.



Proof Sketch: Simultaneous Atomic Density

The limits obtained by fixing one bound and increasing the other are denoted by

$$\mathcal{L}(d) = \lim_{N\to\infty} \alpha(d, N) \quad \text{and} \quad \mathcal{M}(N) = \lim_{d\to\infty} \alpha(d, N).$$

1. Kolekar, Qiu, and Wang showed that $\mathcal{L}(d) \to 1$ as $d \to \infty$ and $\mathcal{M}(N) \to 1$ as $N \to \infty$.
2. Estimates for reducible polynomials give

$$\min\{\mathcal{L}(d), \mathcal{M}(N)\} - \varepsilon(d, N) \leq \alpha(d, N) \leq 1,$$

   where $\varepsilon(d, N) \to 0$ as $d, N \to \infty$ simultaneously.
3. The lower bound approaches 1, so the squeeze theorem gives $\alpha(d, N) \to 1$ as both bounds grow.

Nonuniform Atomic Density

Fix a degree bound $d \geq 1$. For each $0 \leq i \leq d$, the positive integer N_i is the upper bound on the coefficient of x^i. Write $\boldsymbol{N} = (N_0, \ldots, N_d)$. Let $\mathcal{T}(d, \boldsymbol{N})$ count all polynomials $\sum_{i=0}^{d} a_i x^i$ with $0 \leq a_i \leq N_i$, and let $\mathcal{I}(d, \boldsymbol{N})$ count the irreducibles.

$$\alpha(d, \boldsymbol{N}) = \frac{\mathcal{I}(d, \boldsymbol{N})}{\mathcal{T}(d, \boldsymbol{N})}.$$

Theorem (Mittal, Pieleanu, Yan)

For this box, $\alpha(d, \boldsymbol{N}) = \mathcal{I}(d, \boldsymbol{N})/\mathcal{T}(d, \boldsymbol{N})$. As every $N_i \to \infty$,

$$\alpha(d, \boldsymbol{N}) \longrightarrow \frac{1}{\zeta(d+1)}.$$

The constant $1/\zeta(d+1)$ comes from the density of coefficient vectors with $\gcd(a_0, \ldots, a_d) = 1$.

Proof Sketch: Nonuniform Atomic Density

Choose $f = \sum_{i=0}^{d} f_i x^i$ uniformly with $0 \le f_i \le N_i$ for each i. A polynomial is primitive when $\gcd(f_0, \ldots, f_d) = 1$.

1. **Count primitive coefficient vectors.** For fixed d,

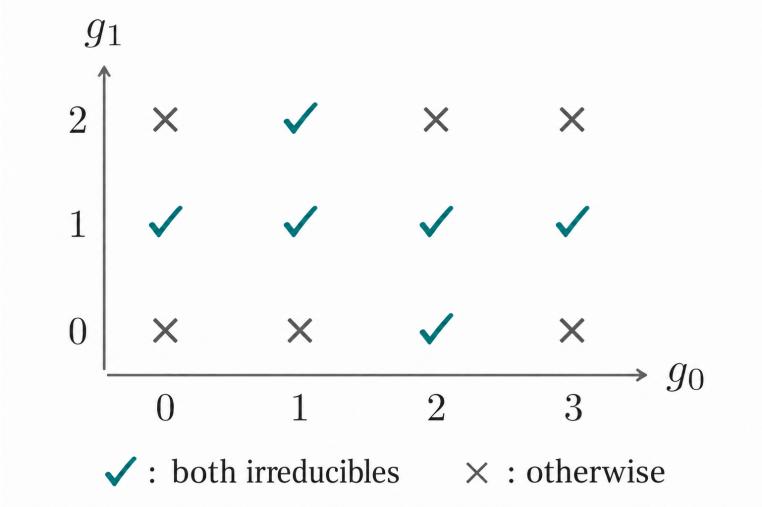
$$\Pr(\gcd(f_0, \ldots, f_d) = 1) \longrightarrow \prod_{p \text{ prime}} (1 - p^{-(d+1)}) = \frac{1}{\zeta(d+1)}.$$

2. **Rational reducibility has density zero.** In fixed degree, the proportion of polynomials reducible over $\mathbb{Q}$ tends to 0 as the coefficient bounds grow.
3. **Conclude.** Every primitive polynomial irreducible over $\mathbb{Q}$ is irreducible in $\mathbb{N}_0[x]$, while every nonprimitive polynomial of positive degree is reducible in $\mathbb{N}_0[x]$. Since rational reducibility has density zero, these bounds yield the desired density.

Goldbach and Goldbach Decompositions

Classical Motivation: Goldbach conjectured that every even integer greater than 2 is a sum of two primes; for example, $28 = 11 + 17$.

Given $f \in \mathbb{N}_0[x]$, we seek $f = g + h$, where g, h are irreducibles in $\mathbb{N}_0[x]$.
For example, $2x + 3 = (x + 2) + (x + 1)$, where both summands are irreducible in $\mathbb{N}_0[x]$. For this example, write $g = g_1 x + g_0$ and $h = f - g$.



Fix $d \geq 1$ and let $f(x) = \sum_{i=0}^{d} f_i x^i$, where $f_i > 0$, and $F = \gcd(f_0, \ldots, f_d)$.

Our question: What fraction of all splits work as f becomes large?

Density of Goldbach Decompositions

Denote by $\mathcal{G}_2(f)$ the number of ordered decompositions $f = g + h$, where both summands are irreducible in $\mathbb{N}_0[x]$. The total number of ordered nonnegative coefficientwise decompositions is $\mathcal{V}_2(f) = \prod_{i=0}^{d}(f_i + 1)$.

Theorem (Mittal, Pieleanu, Yan)

As $\min_i f_i \to \infty$ with d fixed,

$$\frac{\mathcal{G}_2(f)}{\mathcal{V}_2(f)} = \mathfrak{S}_{d+1}(F) + o(1),$$

where

$$\mathfrak{S}_n(F) = \prod_{p|F}(1 - p^{-n}) \prod_{p\nmid F}(1 - 2p^{-n}).$$

Proof Sketch: Goldbach Density

Choose an ordered decomposition $f = g + h$ uniformly. Let $\boldsymbol{g} = (g_0, \ldots, g_d)$ and $\boldsymbol{f} = (f_0, \ldots, f_d)$ denote the coefficient vectors of g and f, respectively. Then, consider a prime p.

1. For the prime p, neither summand is zero modulo p when

$$\boldsymbol{g} \not\equiv \mathbf{0} \pmod{p}, \qquad \boldsymbol{g} \not\equiv \boldsymbol{f} \pmod{p}.$$

   Thus, among the p^{d+1} residue vectors modulo p, one is forbidden if $p \mid F$, and two are forbidden otherwise.
2. The Chinese Remainder Theorem combines these local conditions over finitely many primes. The contribution from large primes goes to 0.
3. In fixed degree, rationally reducible polynomials have density $o(1)$, so replacing "primitive" by "irreducible" changes the proportion by only $o(1)$.

Multiplying the local factors gives $\mathfrak{S}_{d+1}(F)$.

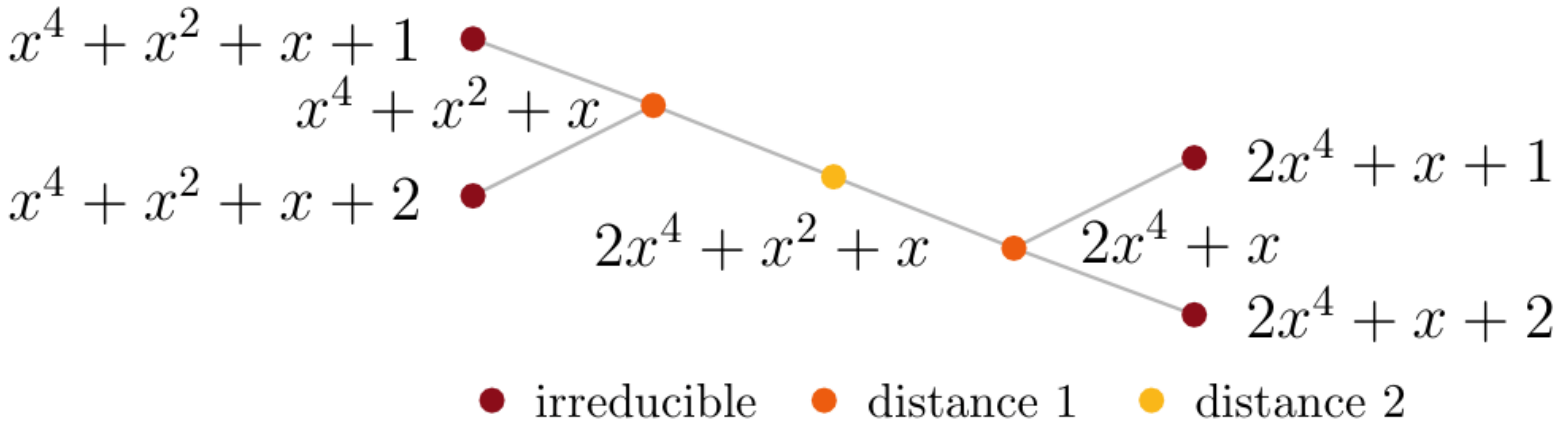
Turán's Conjecture

Turán conjectured that there exists an absolute constant C such that every nonconstant $f \in \mathbb{Z}[x]$ admits a polynomial $g \in \mathbb{Z}[x]$, irreducible over $\mathbb{Q}$, with

$$\deg g \leq \deg f, \qquad \|f - g\|_1 \leq C.$$

Here, the coefficient ℓ_1-distance is the sum of absolute coefficient differences. In particular, distance 1 means changing exactly one integer coefficient by 1.

We consider this question with both $f, g \in \mathbb{N}_0[x]$.



Low Degree Results

For every f in either family below, there exists a monic $g \in \mathbb{N}_0[x]$, irreducible over $\mathbb{Q}$, such that

$$\deg g = \deg f, \qquad \|f - g\|_1 \leq 1.$$

1. Monic cubics with nonzero constant term: $f(x) = x^3 + ax^2 + bx + c$, where $a, b \in \mathbb{N}_0$ and $c \geq 1$.
2. Monic quartics with nonzero constant term whose x^3 coefficient is 0 or 1: $f(x) = x^4 + ax^3 + bx^2 + cx + k$, where $a \in \{0, 1\}$, $k \geq 1$, and $b, c \in \mathbb{N}_0$.

Examples

- $x^3 + x^2 + x + 1 \longmapsto x^3 + x^2 + 2x + 1$,
- $x^4 + 2x^2 + 1 \longmapsto x^4 + 2x^2 + x + 1$

General Degree Result

For polynomials $f = \sum_{i=0}^{n} f_i x^i$ and $g = \sum_{i=0}^{n} g_i x^i$, the coefficient ℓ_1-distance is defined as

$$\|f - g\|_1 = \sum_{i=0}^{n} |f_i - g_i|.$$

Theorem (Mittal, Pieleanu, Yan)

For every degree n polynomial $f \in \mathbb{N}_0[x]$ with $n \geq 2$, there is a degree n polynomial $g \in \mathbb{N}_0[x]$, irreducible over $\mathbb{Q}$, with

$$\|f - g\|_1 \leq 2 + \left\lceil \frac{n-1}{2} + \sqrt{\frac{(n-1)\ln(2n)}{2}} \right\rceil.$$

In particular, the bound is $\frac{n}{2} + O(\sqrt{n \log n})$.

Proof Sketch: General Degree Bound

Set $m = n - 1$ and record the parities of the interior coefficients of f as a vector $\boldsymbol{v} \in \{0, 1\}^m$.

1. **Irreducible binary polynomials are abundant.** The finite field $\mathbb{F}_2 = \{0, 1\}$ has arithmetic modulo 2. There are many monic irreducible polynomials of degree n over $\mathbb{F}_2$.
2. **One irreducible pattern is close to $\boldsymbol{v}$.** A Hamming ball estimate shows that one such parity pattern differs from $\boldsymbol{v}$ in roughly half of the positions.
3. **Lift the pattern to $\mathbb{N}_0[x]$.** Increase those interior coefficients by 1 and adjust the leading and constant coefficients if needed.

The reduction modulo 2 is irreducible, hence the lifted polynomial is irreducible over $\mathbb{Q}$.

Bounded Interior Coefficients

Write $f(x) = \sum_{i=0}^{n} f_i x^i \in \mathbb{N}_0[x]$, where $n \geq 1$, and $B \geq 0$ is an integer.

- **Assumption:** $f_0, f_n \geq 1$ and $f_i \leq B$ for $1 \leq i < n$.
- **Result:** $g = f + (B^2 + 1)$ is irreducible in $\mathbb{R}_{\geq 0}[x]$ (hence in $\mathbb{Q}_{\geq 0}[x]$), and $\|f - g\|_1 = B^2 + 1$, independent of n.
- **Special case:** For interior coefficients in $\{0, 1\}$, $f + 2$ is irreducible, at distance 2.

Example ($B = 1$)

$$f = x^3 + x^2 + x + 1 \quad \longmapsto \quad g = x^3 + x^2 + x + 3$$

$\mathbb{R}_{\geq 0}[x]$ denotes polynomials with nonnegative real coefficients.

Only Logarithmically Many Large Coefficients

Consider $f = \sum_{i=0}^{n} f_i x^i \in \mathbb{N}_0[x]$ of degree $n \geq 1$. Write $e(f) = \#\{0 \leq i \leq n : f_i \geq 2\}$.

- **Hypothesis:** $e(f) \leq \frac{1}{2} \log_2 n$.
- **Result:** There is a degree-n polynomial $g \in \mathbb{N}_0[x]$, irreducible in $\mathbb{Q}_{\geq 0}[x]$, with $\|f - g\|_1 \leq 4098$.
- **For $n \geq 4096$:** The bound improves to 9. We can choose $g = 2x^j + 2x^i + c + f$ for some $0 \leq c \leq 5$ and $1 \leq i < j < n$.

Example

For

$$f(x) = (x^{2000} + 1)\left(x^{2096} + 80x^{1200} + 35x^{500} + 12x^{100} + \sum_{k=0}^{40} x^k\right),$$

we have $e(f) = 6$. Thus, the distance-9 result applies despite the presence of several large coefficients.

Main Results and an Open Problem

Global abundance. We proved that atomic density tends to 1 as degree and coefficient bound both grow, and to $1/\zeta(d+1)$ for fixed degree d in expanding rectangular coefficient boxes.

Additive representations. For large targets, the proportion of successful two irreducible splits is given by the product $\mathfrak{S}_{d+1}(F)$, which depends on the coefficient gcd F.

Local spacing. We proved that a distance of 1 suffices for certain cubic and quartic families, and explicit bounds in general degree.

Main open question: Turán's Conjecture in the nonnegative setting. Can the degree-dependent bound be replaced by an absolute constant while requiring irreducibility over $\mathbb{Q}$?

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