Multiplicative Structure of Rational Monogenic Semidomains

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- $(\mathbb{N}, \cdot)$, where $\mathbb{N} = \{1, 2, 3, \dots\}$
- $(\mathbb{N}_0[x] \setminus \{0\}, \cdot)$, where

$$\mathbb{N}_0[x] = \left\{ \sum_{i=0}^{n} a_i x^i : n \in \mathbb{N}_0, a_i \in \mathbb{N}_0 \right\}.$$

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Question

For which $q \in \mathbb{Q}$ is $(\mathbb{N}_0[q] \setminus \{0\}, \cdot)$ atomic?

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Definition. For $q \in \mathbb{Q}$, we define

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3. If $q = 1/b \in \mathbb{N}^{-1}$ (the set of positive unit fractions) then $S_q^* = \{c/b^n : c, n \in \mathbb{N}\}$.

Monoid of Technical Fractions

Definition. For $q = a/b \in \mathbb{Q}_{>0}$ with $\gcd(a, b) = 1$, let S_q be the rational monogenic semidomain generated by q. We define the technical monoid (or monoid of technical fractions)

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To what extent does T_q influence the structure of S_q?

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  - However, $\frac{5}{4} \notin \langle \frac{1}{p} : p \in \mathbb{P} \rangle$.

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Observation. This proves the existence of non-atomic rational monogenic semidomains.

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Question

For which $q \in \mathbb{Q}_{>0}$ is S_q^* MCD (or MCD-finite)?

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Example. Let $q \in \mathbb{Q}_{>0} \setminus (\mathbb{N} \cup \mathbb{N}^{-1})$. The set

$$\{1 + q + \cdots + q^{329}, (1 + q^3)(1 + q^5)(1 + q^{11})\}$$

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Observation. This construction can be generalized to obtain finite subsets with arbitrarily many distinct MCDs, even if the monoid S_q^* is MCD-finite.

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- Given a monoid M, all atoms and units of M satisfy the ACCP.

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Suppose that

- $q \in (0, 1/2)$,
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Then S_q^* satisfies the quasi-ACCP if and only if it is MCD. Furthermore, if $\inf(\mathcal{A}(T_q)) > 1$, where we define $\inf \varnothing = +\infty$, then S_q^* satisfies neither property.

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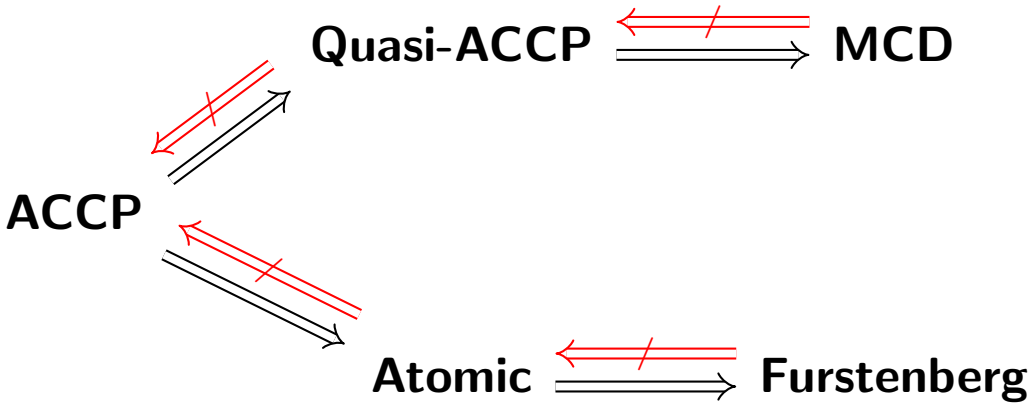
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Conjecture

Let $q = a/b \in \mathbb{Q}_{>0}$ such that $q \in (0, 1/2)$, $a > 1$, and $b > 1$ is not a prime power. Then the set $\mathcal{A}(T_q)$ is finite.

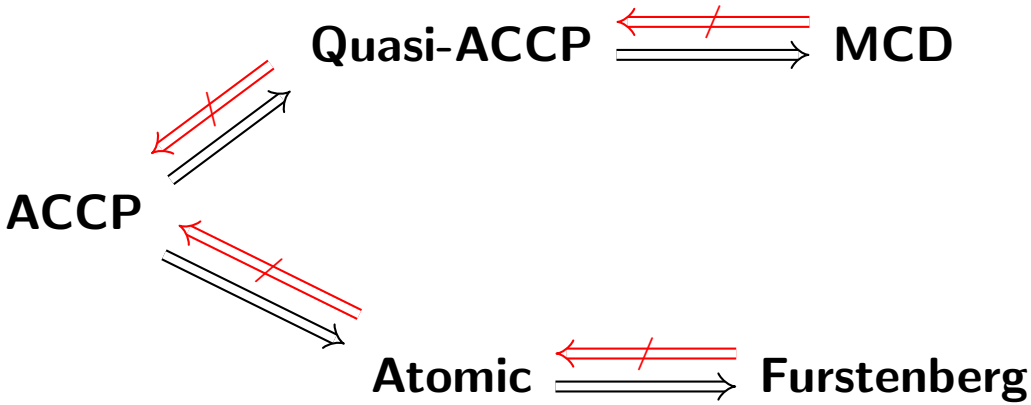
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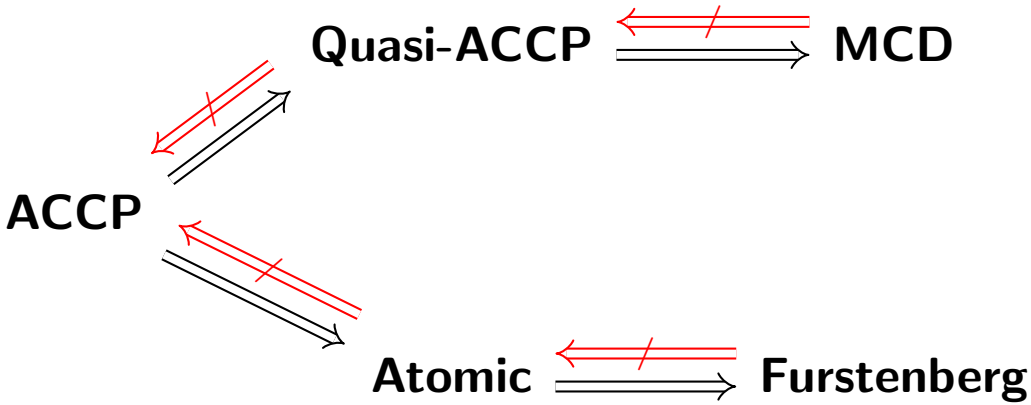
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Remark. For $q \in \mathbb{Q}_{>0}$ with $q \geq 1$, all five properties hold. When $q \in (0, 1/2)$, the quasi-ACCP and MCD properties are equivalent.

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- **Dr. Felix Gotti** for mentoring the project, guiding our research, and providing us with invaluable advice, as well as
- **PRIMES-USA** for giving us this opportunity and making this collaboration possible!

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End of Presentation

Thank you for your attention!