A Topological Interpretation of Divisibility

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Rings

Definition

A **ring** is a set R equipped with two binary operations $+$ and $\times$ such that:

1. $(R, +)$ is an abelian group.
2. Multiplication is commutative and associative.
3. Multiplication distributes over addition.

If there exists $1 \in R$ such that $1 \times a = a \times 1 = a$ for all $a \in R$, then we call 1 the **multiplicative identity**, and R is **unital**.

Definition

An **integral domain** is a nonzero ring such that $ab = 0$ implies $0 \in \{a, b\}$.

Standard Definitions

Let R be a ring.

- We say x **divides** y and denote $x \mid y$ if there exists $r \in R$ such that $y = xr$.
- A subset $S \subseteq R$ is an **ideal** if S is an additive subgroup of R and $rs \in S$ for all $r \in R$ and $s \in S$.
- The ideal generated by elements $\{a_1, a_2, \ldots, a_n\} \subseteq R$, denoted $(a_1, a_2, \ldots, a_n)$, is the ideal
$$\left\{ \sum_{i=1}^{n} r_i a_i : r_i \in R \right\}.$$
- An ideal S is **principal** if $S = (r)$ for some $r \in R$.
- Two elements $a, b \in R$ are **associates** if $(a) = (b)$. We denote this as $a \sim b$.
- Suppose R is unital. A **unit** u of a ring R is an element such that there exists $v \in R$ such that $uv = 1$.

Topology

In this research, we study rings through the topological lens.

Definition

Let X be a set. A **topology** on X is a collection τ of subsets of X, whose elements are called **open sets**, such that:

1. $\varnothing \in \tau$ and $X \in \tau$;
2. If $\{U_i\}_{i \in I}$ is any collection of sets in τ, then $\bigcup_{i \in I} U_i \in \tau$;
3. If $U_1, \ldots, U_n \in \tau$, then $\bigcap_{i=1}^{n} U_i \in \tau$.

A topology is **Alexandrov** if the intersection of arbitrary families of open sets is open.

Example

The set $X = \{a, b, c\}$ with $\tau = \{\varnothing, \{a\}, \{a, b\}, X\}$ is an Alexandrov topology.

Prior Research

Topological methods have long been used to encode algebraic structure.

Example

The Zariski topology studies a ring through its prime ideals.

In 2025, Yiğit and Koç introduced an Alexandrov topology related to divisibility, using associate classes of nonzero nonunits in integral domains.

Motivating Question

How can divisibility be encoded in a topological framework?

SMC Sets

To encode divisibility in a topological framework, we consider the following.

Definition

Let R be a ring. A subset $S \subseteq R$ is said to be **strongly multiplicatively closed (SMC)** if $r \in R$ and $s \in S$ implies $rs \in S$.

These sets are also known as **semigroup ideals** and were studied by Olberding (2010).

Example

In $\mathbb{Z}$, the sets (2) and $(2) \cup (3)$ are SMC.

Proposition

Let R be a unital ring. A subset $S \subseteq R$ is strongly multiplicatively closed if and only if it is a union of principal ideals.

The Topology

We consider the following topology for this research.

Definition

The **strongly multiplicatively closed (SMC) topology** on a ring R is the topology whose open sets are its SMC subsets.

Example

The SMC topology on $\mathbb{Z}/4\mathbb{Z}$ has exactly four open sets: $\varnothing$, $\{0\}$, $\{0, 2\}$, and $\mathbb{Z}/4\mathbb{Z}$.

The SMC topology is governed by divisibility.

Properties

Let R be a unital ring and let $x, y \in R$.

- The smallest open set containing an element x is the ideal (x).
- The element x divides y if and only if every open set containing x also contains y.

Example

Consider the SMC topology on $\mathbb{Z}$.

- The smallest open set containing the element 6 is (6).
- Let U be an open set such that $\{2, 3\} \cap U \neq \varnothing$. Then $6 \in U$.

Compactness of Rings

The SMC topology is Alexandrov, making compactness a natural property to investigate.

Definition

Let (R, τ) be a topological space. A subset $K \subseteq R$ is said to be **compact** if every open cover of K admits a finite subcover.

We can characterize precisely which rings are compact under the SMC topology.

Theorem (Gupta-Wu-Zhan, 2026)

Let R be a ring. Then R is compact under the SMC topology on R if and only if R is finite or unital.

Examples

Compact

The ring $\mathbb{Z}$ is compact in the SMC topology.

Proof. Let $\{U_i\}_{i\in I}$ be an open cover of $\mathbb{Z}$. Then $1 \in U_i$ for some $i \in I$. Since U_i is SMC, $\mathbb{Z} = (1) \subseteq U_i$, so $\{U_i\}$ is a finite subcover.

Not Compact

The ring $2\mathbb{Z}$ is not compact in the SMC topology.

Proof. For each odd integer i, let $U_i = 4\mathbb{Z} \cup \{2i\}$. These SMC sets cover $2\mathbb{Z}$, but $2i$ belongs only to U_i. Thus no finite subcollection covers $2\mathbb{Z}$.

Compact Ideals in Rings

For the remainder of the presentation, all rings are assumed to be unital.
Our topology can be equivalently defined by taking the set of principal ideals as its open basis.
The following condition then determines compactness for ideals under the SMC topology.

Proposition

In the SMC topology, an ideal is compact if and only if it is a finite union of principal ideals.

It follows that all principal ideals are compact.

We find rings in which the compact ideals are precisely the principal ideals.

Rings in Which Compact Ideals Are Principal

Definition

A **u-ring** R is a ring such that any ideal in R which is contained in a finite union of ideals of R must be entirely contained within one of those ideals.

A result of Quartararo and Butts (1975) on u-rings implies that, for each of the following classes of rings, the compact ideals in the SMC topology are precisely the principal ideals:

- Rings containing an infinite field (nonzero commutative ring in which every nonzero element has a multiplicative inverse).
- **Bézout rings** — rings in which for all $a, b \in R$, there exists $c \in R$ such that $(a, b) = (c)$.

Example

- The rings $\mathbb{Q}$, $\mathbb{Q}[x]$, and $\mathbb{Q} + x\mathbb{R}[x]$ satisfy the first condition.
- The ring $\mathbb{Z}$ is a Bézout ring since every ideal in $\mathbb{Z}$ is principal.

Example of a Ring With a Compact Non-Principal Ideal

A ring with a compact, non-principal ideal:

Example

The ring $\mathbb{F}_2[x,y]/(x^2,xy,y^2)$ does not satisfy the property that all compact ideals are principal.

Proof. We have $(\overline{x},\overline{y}) = (\overline{x}) \cup (\overline{y}) \cup (\overline{x+y})$. Thus $(\overline{x},\overline{y})$ is a compact, non-principal ideal under the SMC topology on the ring $\mathbb{F}_2[x,y]/(x^2,xy,y^2)$.

Definitions

Let R be a ring.

Definition

A nonzero nonunit $\pi \in R$ is **irreducible** if $\pi = ab$ implies that $\pi \sim a$ or $\pi \sim b$.

Example

In the ring $\mathbb{Z}$, all prime numbers are irreducibles.

Definition

A **Unique Factorization Domain (UFD)** is a domain in which every nonzero, nonunit element can be factored uniquely into a product of irreducible elements up to associates.

Unique Factorization Domains

Theorem (Gupta-Wu-Zhan, 2026)

If R is a UFD, all compact ideals are principal.

Example

Every compact ideal in $\mathbb{Z}[x]$ is principal.

However, $\mathbb{Z}[x]$ is not a u-ring, since $(2, x) = (2, x^2) \cup (4, x) \cup (4, x + 2)$.

Product of Rings

Definition

The **direct product** of rings $R_1, R_2, \cdots, R_n$, denoted by $R_1 \times \cdots \times R_n$, is the Cartesian product of the underlying sets of rings R_i equipped with component-wise addition and multiplication.

Theorem (Gupta-Wu-Zhan, 2026)

If $R = R_1 \times R_2 \times \cdots \times R_n$ is a direct product of rings, then every compact ideal of R is principal if and only if the same property holds for every R_i.

Example

In the ring $\mathbb{Q}[x, y] \times \mathbb{Z}[x]$, all compact ideals are principal.

Proof. Note that $\mathbb{Q}[x, y]$ and $\mathbb{Z}[x]$ are both UFDs, so all compact ideals in each of these rings are principal. The result follows from the preceding theorem.

Noetherian

Definition

Let R be a ring. The ring R is **Noetherian** if every ascending chain of ideals stabilizes.
Let (X, τ) be a topological space. The topology τ is **Noetherian** if every ascending chain of open sets stabilizes.

Definition

A **Cohen–Kaplansky** ring is a Noetherian ring with finitely many irreducibles up to associates.

We proved the following theorem.

Theorem (Gupta-Wu-Zhan, 2026)

Let R be a ring. Its SMC topology is Noetherian if and only if R is a Cohen–Kaplansky ring.

Examples

Cohen–Kaplansky

The ring $R = \mathbb{Z}_{(2)} = \{a/b : a, b \in \mathbb{Z},\ b \text{ odd}\}$ is a Cohen–Kaplansky ring.

Proof. The open sets are $\varnothing$, $\{0\}$, and (2^n) for $n \geq 0$. Note that $(2^n) \subsetneq (2^m)$ if and only if $n > m$. Hence no infinite strictly ascending chain of open sets exists.

Not Cohen–Kaplansky

The ring $\mathbb{Z}$ has infinitely many irreducibles up to associates, so it is not a Cohen–Kaplansky ring.

Proof. For distinct positive primes $p_1, p_2, \ldots$, let $V_n = \bigcup_{i=1}^{n} (p_i)$. Therefore $V_1 \subsetneq V_2 \subsetneq \ldots$. Each V_n is SMC, hence open, and $p_{n+1} \in V_{n+1} \setminus V_n$, so this chain never stabilizes.

Artinian

Definition

Let R be a ring. The ring R is **Artinian** if every descending chain of ideals stabilizes.
Let (X, τ) be a topological space. The topology τ is **Artinian** if every descending chain of open sets stabilizes.

We proved the following theorem.

Theorem (Gupta-Wu-Zhan, 2026)

Let R be a ring. Then its SMC topology is Artinian if and only if R is an Artinian ring with finitely many irreducibles up to associates.

Examples

Artinian

The ring $\mathbb{Z}/4\mathbb{Z}$ is finite, so it is Artinian and contains only finitely many irreducibles up to associates.

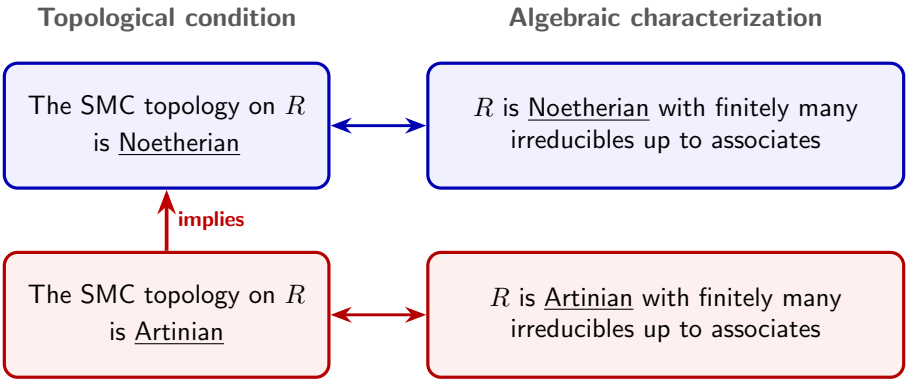
Proof. Its SMC topology has only finitely many open sets. Therefore, every descending chain of open sets stabilizes.

Not Artinian

The ring $\mathbb{Z}$ is not Artinian.

Proof. For each $n \geq 1$, let $U_n = (2^n)$. Each U_n is SMC, hence open, and $U_1 \supsetneq U_2 \supsetneq \dots$, so this descending chain never stabilizes.

Noetherian vs. Artinian



Other Results

- We extend the preceding results on chain conditions and compactness to semigroups.
- We study continuous maps and homeomorphisms between rings equipped with our topology.
- We fully characterize Noetherian polynomial u-rings and characterize certain compact ideals in polynomial rings.

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Questions?