A Combinatorial Bijection Between Orbits on Flags

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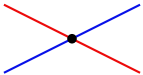
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Two lines in a plane

What are the possible configurations of two lines in a plane?



Parallel



Intersect

Equal

Two lines in $\mathbb{R}^3$

What about in 3 dimensions?

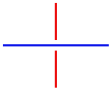


Parallel



Intersect

Equal

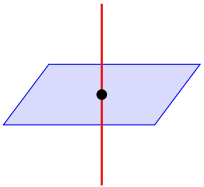


Skew

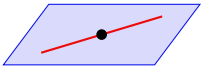
Skew lines never meet, and they are not parallel.

A line and a plane through the origin of $\mathbb{R}^3$

From now on, all subspaces pass through the origin.



Meet only at the origin



Line is contained

In $\mathbb{R}^n$ for any $n \geq 3$, these are the only two possibilities.

General Arrangements

Definition

The Grassmannian $\mathrm{Gr}(k, \mathbb{R}^n)$ is the set of k-dimensional subspaces of $\mathbb{R}^n$.

- $U \in \mathrm{Gr}(u, \mathbb{R}^n)$
- $W \in \mathrm{Gr}(w, \mathbb{R}^n)$

How can U and W sit inside $\mathbb{R}^n$?

Example

A line U and a plane W in $\mathbb{R}^3$ is the case $(u, w) = (1, 2)$ and $n = 3$.

General Arrangements

Two arrangements are "the same" if a change of coordinates takes one to the other.

Definition

Define equivalence relation $\sim$ by

$$(U, W) \sim (U', W') \iff (U', W') = (gU, gW) \text{ for some } g \in \mathrm{GL}_n(\mathbb{R}).$$

The equivalence classes of $\sim$ are called $\mathrm{GL}_n(\mathbb{R})$-orbits.

What are the $\mathrm{GL}_n(\mathbb{R})$-orbits of $\mathrm{Gr}(u, \mathbb{R}^n) \times \mathrm{Gr}(w, \mathbb{R}^n)$?

General Arrangements

Look at $\dim(U \cap W)$.

- An invertible change of coordinates does not change this number.

Fact

The orbit of (U, W) is determined by $\dim(U \cap W)$.

Example

For a line and a plane in $\mathbb{R}^3$, $(u, w) = (1, 2)$ and $n = 3$:

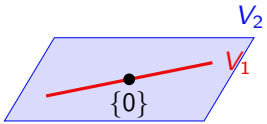
- $\dim(U \cap W) = 0$: the line meets the plane only at the origin.
- $\dim(U \cap W) = 1$: the line is contained in the plane.

These are exactly the two pictures from before.

Flags

Definition

- A flag is a chain of nested subspaces $\{0\} \subset V_1 \subset V_2 \subset \cdots \subset V_k = \mathbb{R}^n$.
- A complete flag is a flag $\{0\} \subset V_1 \subset V_2 \subset \cdots \subset V_n = \mathbb{R}^n$ where $\dim V_i = i$.
- The complete flag variety $\mathrm{Fl}_n(\mathbb{R})$ is the set of all complete flags in $\mathbb{R}^n$.



A complete flag in $\mathbb{R}^3$:
point $\subset$ line $\subset$ plane $\subset \mathbb{R}^3$

Example

The standard flag in $\mathbb{R}^n$ is $\{0\} \subset \mathrm{span}(e_1) \subset \mathrm{span}(e_1, e_2) \subset \cdots \subset \mathrm{span}(e_1, \ldots, e_n)$.

Adapted Bases

Definition

A basis $v_1, \ldots, v_n$ of $\mathbb{R}^n$ determines the complete flag

$$\mathcal{F}(v_1, \ldots, v_n) := \{0\} \subset \operatorname{span}(v_1) \subset \operatorname{span}(v_1, v_2) \subset \cdots \subset \operatorname{span}(v_1, \ldots, v_n).$$

We call $(v_1, \ldots, v_n)$ an adapted basis of this flag.

- Every complete flag has an adapted basis: pick $v_i \in V_i \setminus V_{i-1}$.
- Adapted bases are not unique: $\mathcal{F}(e_1, e_2, e_3) = \mathcal{F}(e_1, e_1 + e_2, e_3)$.

Example

The standard flag in $\mathbb{R}^n$ is $\mathcal{F}(e_1, \ldots, e_n)$.

Flags

What are the $\mathrm{GL}_n(\mathbb{R})$-orbits of $\mathrm{Fl}_n(\mathbb{R}) \times \mathrm{Fl}_n(\mathbb{R})$?

Theorem (Bruhat Decomposition)

The $\mathrm{GL}_n(\mathbb{R})$-orbits of $\mathrm{Fl}_n(\mathbb{R}) \times \mathrm{Fl}_n(\mathbb{R})$ are in bijection with the symmetric group S_n.

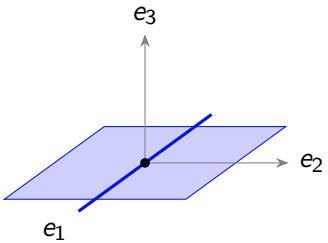
A representative of the $\mathrm{GL}_n(\mathbb{R})$-orbit corresponding to $\sigma \in S_n$ is

$$\big(\mathcal{F}(e_1, \ldots, e_n),\ \mathcal{F}(e_{\sigma(1)}, \ldots, e_{\sigma(n)})\big).$$

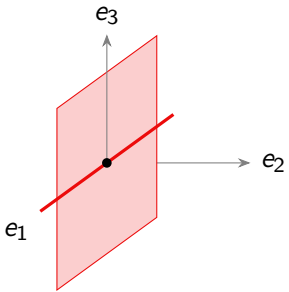
Flags

Example

$n = 3$ and the transposition $\sigma = (2\ 3)$, so $\sigma(1) = 1$, $\sigma(2) = 3$, $\sigma(3) = 2$.



$\mathcal{F}$: $\quad \text{span}(e_1) \subset \text{span}(e_1, e_2)$



$\mathcal{F}_{(2\ 3)}$: $\quad \text{span}(e_1) \subset \text{span}(e_1, e_3)$

A flag and two complementary subspaces

Now consider triples $(\mathcal{F}, U, W)$ in $\mathbb{R}^{p+q}$ where

- $\mathcal{F} \in \mathrm{Fl}_{p+q}(\mathbb{R})$ is a complete flag,
- $U \in \mathrm{Gr}(p, \mathbb{R}^{p+q})$ and $W \in \mathrm{Gr}(q, \mathbb{R}^{p+q})$,
- $U \cap W = \{0\}$, so $U \oplus W = \mathbb{R}^{p+q}$.

That is, the $\mathrm{GL}_{p+q}(\mathbb{R})$-orbits under $g \cdot (\mathcal{F}, U, W) = (g\mathcal{F}, gU, gW)$.

Example

$p = q = 1$: the flag is just a line V_1 in the plane, and U, W are two distinct lines. Three $\mathrm{GL}_2(\mathbb{R})$-orbits: $V_1 = U$, $V_1 = W$, or V_1 is neither.

Clans

Definition (Matsuki–Oshima)

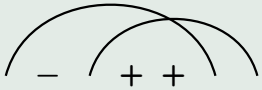
A (p, q)-clan is a way to

- pair up some of the vertices $\{1, \ldots, p+q\}$, and
- label each remaining vertex by a "$+$" or "$-$",

such that $\#\{+\} - \#\{-\} = p - q$.

Example

A $(4, 3)$-clan on $\{1, \ldots, 7\}$:



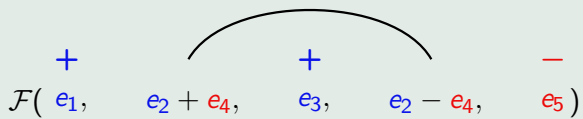
A flag and two complementary subspaces

Theorem (Matsuki–Oshima)

The $\mathrm{GL}_{p+q}(\mathbb{R})$-orbits of $(\mathcal{F}, U, W)$ with $\dim U = p$, $\dim W = q$, $U \cap W = \{0\}$ are in bijection with (p, q)-clans.

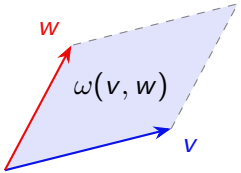
Example

Example for $(3, 2)$, with $U = \mathrm{span}(e_1, e_2, e_3)$ and $W = \mathrm{span}(e_4, e_5)$:



Signed area

How do we measure the signed area of the parallelogram spanned by $v, w \in \mathbb{R}^2$?



$$\omega(v, w) = v_1 w_2 - v_2 w_1 = \det \begin{pmatrix} v_1 & w_1 \\ v_2 & w_2 \end{pmatrix}$$

- Positive if w is counterclockwise from v, negative if clockwise.
- Swapping v and w flips the sign: $\omega(w, v) = -\omega(v, w)$.

Symplectic Forms

A symplectic form is an abstract version of signed area.

Definition

A symplectic form ω is a map $\omega\colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ which is

- *Bilinear*: linear in each argument.
- *Skew-symmetric*: $\omega(v, w) = -\omega(w, v)$ for all $v, w \in \mathbb{R}^n$. In particular, $\omega(v, v) = 0$.
- *Nondegenerate*: for each nonzero $v \in \mathbb{R}^n$, there exists a $w \in \mathbb{R}^n$ for which $\omega(v, w) \neq 0$.

Example

The standard symplectic form ω_0 on $\mathbb{R}^{2n}$ is

$$\omega_0(v, w) = v^T \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} w = \sum_{i=1}^{n} \left(v_i \, w_{n+i} - v_{n+i} \, w_i \right).$$

ω_0 adds up the signed areas of the projections onto the n coordinate planes $\operatorname{span}(e_i, e_{n+i})$. When $n = 1$, it is exactly the signed area from before.

Symplectic Forms

A space equipped with a symplectic form must have even dimension.

Definition

Let ω be a symplectic form on $\mathbb{R}^{2n}$. A symplectic basis is a basis $a_1, \ldots, a_n, b_1, \ldots, b_n$ such that

$$\omega(a_i, b_i) = 1, \quad \omega(a_i, b_j) = 0 \text{ for } i \neq j, \quad \text{and} \quad \omega(a_i, a_j) = \omega(b_i, b_j) = 0$$

for all $1 \leq i, j \leq n$.

Fact

Every symplectic form on $\mathbb{R}^{2n}$ has a symplectic basis.

Example

The standard basis $e_1, \ldots, e_n, e_{n+1}, \ldots, e_{2n}$ is a symplectic basis of ω_0.

A flag and a symplectic form

Consider pairs $(\mathcal{F}, \omega)$ where $\mathcal{F} \in \mathrm{Fl}_{2n}(\mathbb{R})$ and ω is a symplectic form on $\mathbb{R}^{2n}$.

$\mathrm{GL}_{2n}(\mathbb{R})$ acts on symplectic forms by change of coordinates:

$$(g \cdot \omega)(v, w) = \omega(g^{-1}v,\, g^{-1}w).$$

That is, the $\mathrm{GL}_{2n}(\mathbb{R})$-orbits under $g \cdot (\mathcal{F}, \omega) = (g\mathcal{F}, g \cdot \omega)$.

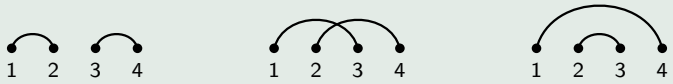
Perfect Matchings

Definition

A perfect matching on $\{1, \ldots, 2n\}$ is a way to split the vertices into n pairs.

Example

The three perfect matchings on $\{1, 2, 3, 4\}$:



There are $(2n-1)!!$ perfect matchings on $\{1, \ldots, 2n\}$.

A flag and a symplectic form

Theorem (Matsuki)

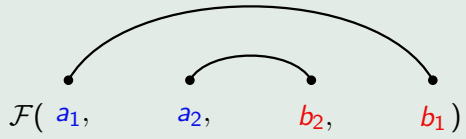
The $\mathrm{GL}_{2n}(\mathbb{R})$-orbits of $(\mathcal{F}, \omega)$ are in bijection with perfect matchings on $\{1, \ldots, 2n\}$.

The flag can always be written as

$$\mathcal{F}: \ \{0\} \subset \mathrm{span}(v_1) \subset \mathrm{span}(v_1, v_2) \subset \cdots \subset \mathrm{span}(v_1, \ldots, v_{2n})$$

where $v_1, \ldots, v_{2n}$ is a symplectic basis for ω, in some order. The matching records which pairs v_i, v_j satisfy $\omega(v_i, v_j) = \pm 1$.

Example for $n = 2$



A flag and a symplectic hyperplane

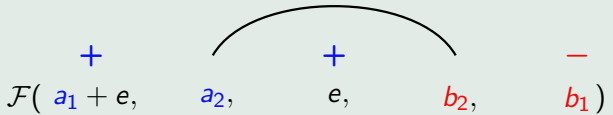
Now consider $(\mathcal{F}, H, \omega, e)$ in $\mathbb{R}^{2n+1}$, where $\mathcal{F} \in \mathrm{Fl}_{2n+1}(\mathbb{R})$, H is a hyperplane (dimension $2n$) with a symplectic form ω on H, and $e \notin H$ is a vector.

Theorem (Matsuki; independently by L.)

The $\mathrm{GL}_{2n+1}(\mathbb{R})$-orbits of $(\mathcal{F}, H, \omega, e)$ are in bijection with $(n+1, n)$-clans.

Example

Example for $n = 2$, where a_1, a_2, b_1, b_2 is a symplectic basis of H and $e \notin H$:



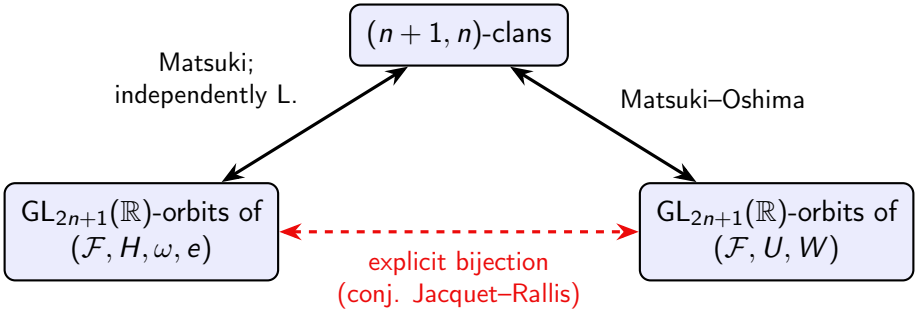
A Conjecture

Conjecture (special case of Jacquet–Rallis)

In $\mathbb{R}^{2n+1}$, there is an explicit bijection between the $\mathrm{GL}_{2n+1}(\mathbb{R})$-orbits of

- $(\mathcal{F}, U, W)$ with $\dim U = n+1$, $\dim W = n$, $U \cap W = \{0\}$, and
- $(\mathcal{F}, H, \omega, e)$ with ω a symplectic form on a hyperplane H and $e \notin H$.

The Bijection



Example for $n = 1$

$$\mathcal{F}(a_1 + e,\ e,\ b_1) \longleftrightarrow\ +\ +\ -\ \longleftrightarrow \mathcal{F}(e_1,\ e_2,\ f_1)$$

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The End

Thank you!!