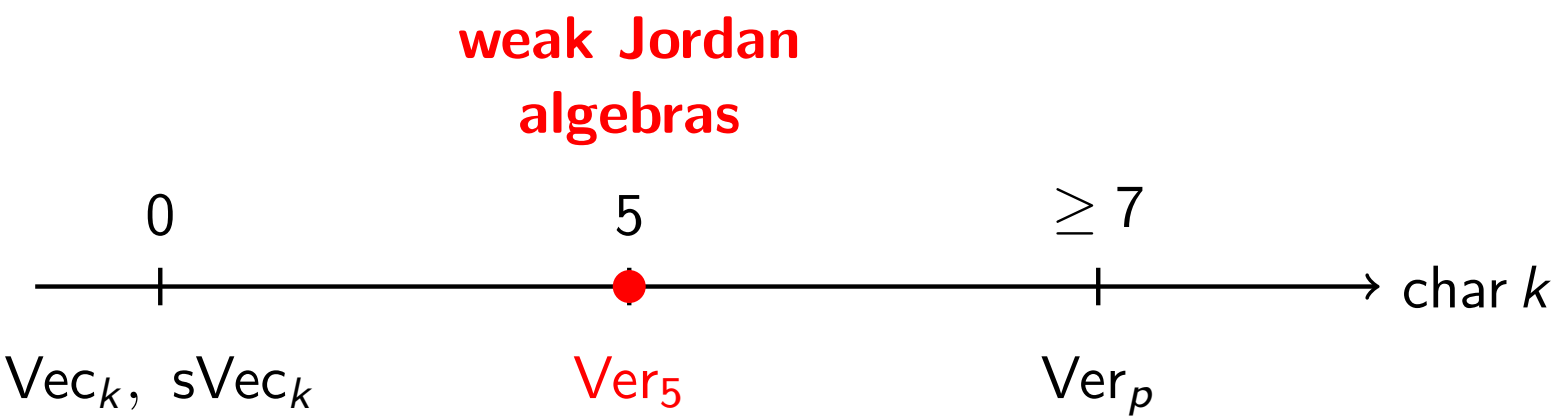
Weak Jordan Algebras from the Verlinde Category in Characteristic 5

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Motivation


weak Jordan algebras
0
5
≥ 7
char k
Vec_k, sVec_k
Ver_5
Ver_p

Representations of $\mathbb{Z}/p$ in characteristic p

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Example

$V = k^3$ and $\mathbb{Z}/p$-action is $1 \cdot v = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} v$. Well-defined and indecomposable if $p \geq 3$.

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- "Freshman's dream" is satisfied: $(g - \mathbb{I})^p = g^p - \mathbb{I}^p = \mathbb{I} - \mathbb{I} = 0$.
- Indecomposable representations are single Jordan blocks

$$V_1, V_2, \ldots, V_p, \qquad \dim V_i = i.$$

Semisimplification and the Verlinde category

Symmetric tensor category $\mathrm{Rep}(\mathbb{Z}/p)$ is additively generated by $V_1, \ldots, V_p$. Objects are vector spaces with $\mathbb{Z}/p$-action. Morphisms are $\mathbb{Z}/p$-equivariant linear maps.

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Semisimplifying symmetric tensor category [EO19]

Keep the indecomposables of **nonzero dimension** and throw the others away. The remaining objects become the simple objects.

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Define **Verlinde category** $\mathrm{Ver}_p := \overline{\mathrm{Rep}(\mathbb{Z}/p)}$ by semisimplification. Observe $\dim V_p = p \equiv 0 \bmod p \implies V_p \mapsto 0$. Then Ver_p is additively generated by relabeled survivors:

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Alternatively built from $\mathfrak{sl}_2$ with $L_i \leftrightarrow i$-dim $\mathfrak{sl}_2$-irrep. So L_1 trivial module, L_3 adjoint representation $(\mathfrak{sl}_2 \curvearrowright \mathfrak{sl}_2)$.

Kantor-Koecher-Tits construction in Ver_p

Definition (KKT Lie Algebra in Ver_p)

Lie algebra $\mathfrak{g}$ in Ver_p is a **KKT Lie algebra** if $\mathfrak{g} = (\mathfrak{g}_0 \otimes L_1) \oplus (A \otimes L_3)$, $\mathfrak{g}_0$ and A spaces. **Reduced** if $\mathfrak{g}_0 \curvearrowright A$ is faithful.

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Spaces commute with L_i:

$$\mathfrak{g} \otimes \mathfrak{g} = (\mathfrak{g}_0 \otimes \mathfrak{g}_0) \otimes L_1 \ \oplus \ (\mathfrak{g}_0 \otimes A) \otimes L_3 \ \oplus \ (A \otimes \mathfrak{g}_0) \otimes L_3 \\ \oplus \ (A \otimes A) \otimes (L_3 \otimes L_3)$$

Lie bracket matches $L_1 \to L_1$, $L_3 \to L_3$. Spaces carry the rest of the info:

$\mathfrak{g}_0 \otimes \mathfrak{g}_0$	$\to$	$\mathfrak{g}_0$	Lie bracket $[-,-]$	
$\mathfrak{g}_0 \otimes A$	$\to$	A	action ρ	
$A \otimes A$	$\to$	A	product $*$	**symmetric**
$A \otimes A$	$\to$	$\mathfrak{g}_0$	map b	**skew**

Jordan vs. weak Jordan

Jordan

A commutative algebra, $b : A \otimes A \to \mathrm{Der}(A)$ skew.

1. $x(zy) - (xz)y = b(x, y)z$
2. and $b(x, x^2) = 0$.

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Theorem ([EKO26])

$\mathfrak{g} = (\mathfrak{g}_0 \otimes L_1) \oplus (A \otimes L_3)$ *reduced KKT in* $\mathrm{Ver}_p \Rightarrow \mathfrak{g}_0 \subseteq \mathrm{Der}(A)$ *and*

$p = 0$ *or* $p \geq 7$: A *is Jordan*

$p = 5$: A *is* **weak** *Jordan*

The exceptionals and Magic Square construction

Over $\mathbb{C}$: simple Lie algebras

$$\underbrace{\mathfrak{a}_n\ ,\mathfrak{b}_n,\ \mathfrak{c}_n,\ \mathfrak{d}_n}_{\mathfrak{sl},\mathfrak{so},\mathfrak{sp}} \quad \& \quad \underbrace{\mathfrak{g}_2,\ \mathfrak{f}_4,\ \mathfrak{e}_6,\ \mathfrak{e}_7,\ \mathfrak{e}_8}_{\text{exceptional}}$$

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Freudenthal-Tits Magic Square

C composition algebra, J Jordan algebra:

$$\mathcal{T}(C,J) = \mathrm{Der}(C) \oplus \mathrm{Inder}(J) \oplus (C^0 \otimes J^0).$$

C^0 and J^0 are trace-free parts. E.g. if $C = \mathbb{C}$ over $\mathbb{R}$, $C^0 = \langle i \rangle$.

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$C = \mathbb{O}_s$, $J = H_3(K)$ for K split composition algebra:

	$H_3(\mathbb{R}_s)$	$H_3(\mathbb{C}_s)$	$H_3(\mathbb{H}_s)$	$H_3(\mathbb{O}_s)$
$\mathbb{O}_s$	$\mathfrak{f}_4$	$\mathfrak{e}_6$	$\mathfrak{e}_7$	$\mathfrak{e}_8$

Structurable algebras inside $\mathfrak{g}$

Define **matrix structurable algebra** as in [Med17]

$$A(J) = \left\{ \begin{pmatrix} \alpha & S \\ T & \beta \end{pmatrix} : \alpha, \beta \in k,\ S, T \in J \right\}, \qquad \overline{\begin{pmatrix} \alpha & S \\ T & \beta \end{pmatrix}} = \begin{pmatrix} \beta & S \\ T & \alpha \end{pmatrix}.$$

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Hermitian subspace $H(J) = \{x \in A(J) : \overline{x} = x\}$ is closed under the symmetrized product.

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Theorem ([EO06])

Let $\mathfrak{g} = \mathcal{T}(\mathbb{O}_s, J)$ with $J = H_3(K)$. Suppose $\mathfrak{sl}_2 \subseteq \mathfrak{g}$ such that

$$\mathfrak{g} \cong (\mathfrak{d} \otimes V_1) \oplus (\mathcal{H} \otimes V_3) \oplus (\mathcal{S} \otimes V_5),$$

as $\mathfrak{sl}_2$-modules (V_i irreducible, dimension i). Then $\mathcal{A} = \mathcal{H} \oplus \mathcal{S}$ is an algebra (from $\mathfrak{g}$ Lie bracket) with involution $\overline{h+s} := h - s$, and

$$\boxed{\mathcal{A} \cong A(J)}$$

Putting it together in Ver_5

1. $\mathfrak{g} = \mathcal{T}(\mathbb{O}_s, J),\ J = H_3(K) \implies \quad \mathfrak{g} \in \{\mathfrak{f}_4, \mathfrak{e}_6, \mathfrak{e}_7, \mathfrak{e}_8\}$

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2. Principal $\mathfrak{sl}_2$ (from A_2 root subsystem):

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3. Then

$$\boxed{\mathcal{H} \oplus \mathcal{S} = \mathcal{A} \cong A(J), \qquad \mathcal{H} \cong H(J).}$$

4. When $p = 5$, semisimplification kills V_5 ($\dim V_5 = 5 \equiv 0$)

$$\mathfrak{g} \rightsquigarrow (\mathfrak{d} \otimes L_1) \oplus (\mathcal{H} \otimes L_3),$$

a reduced KKT Lie algebra, so $\mathcal{H}$ is a **weak Jordan algebra**.

Concrete example: the $\mathfrak{f}_4$ case

$J = H_3(k)$ cubic Jordan algebra with

$$N(S) := \det S, \quad S^\sharp := \mathrm{adj}(S), \quad \tau(S, T) := \mathrm{Tr}(ST)$$

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- $x = (\lambda, S, T)$, $y = (\mu, S', T')$ in $H(J)$. Define

$$b(x,y) := \pi(ST' - S'T) \in \operatorname{Der}(H(J)), \quad \pi(Q) := Q - \frac{1}{3}\operatorname{Tr}(Q)\mathbb{I}_3$$

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Result: weak Jordan, simple, not Jordan.

Conclusion: four simple weak Jordan algebras

Lemma (Failure of Jordan identity)

For $\mathfrak{g} \in \{\mathfrak{f}_4, \mathfrak{e}_6, \mathfrak{e}_7, \mathfrak{e}_8\}$, *there exists* $x, y, z \in H(J) \cong \mathcal{H}$ *with:*

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Theorem [Akr26]

Let k be an algebraically closed field of characteristic 5. For $K \in \{\mathbb{R}_s, \mathbb{C}_s, \mathbb{H}_s, \mathbb{O}_s\}$, let $J_K := H_3(K)$, let $\mathfrak{g}_K = \mathcal{T}(\mathbb{O}_s, J_K)$, and let $\mathcal{H}_K := H(J_K) \subseteq A(J_K)$ be the Hermitian subspace. Then $\mathcal{H}_K$ is a weak Jordan algebra. Moreover, $\mathcal{H}_K$ is simple and is not a Jordan algebra.

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