Ricci flow on ALH manifolds

Minkyu Jo

PRIMES Conference

Mentor: Isaac M. Lopez
University of Chicago

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What is a differentiable manifold?

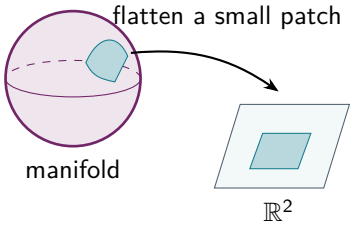
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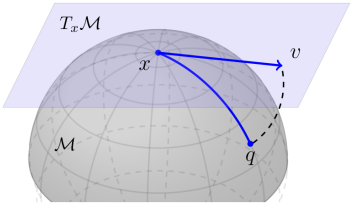
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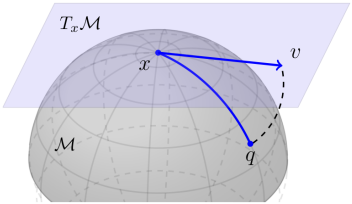


- These small patches fit together smoothly.

Tangent vectors and tangent spaces



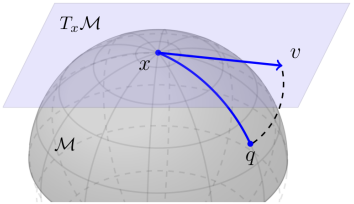
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- The **tangent space** T_xM is the vector space of all tangent vectors at x.

Riemannian Manifolds

Definition (Riemannian Manifold)

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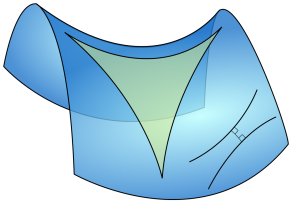
Length of a curve $\gamma : [0, 1] \to M$:

$$L_g(\gamma) = \int_0^1 \sqrt{g_{\gamma(t)}(\gamma'(t), \gamma'(t))}\, dt.$$

Hyperbolic space

Definition (Hyperbolic space)

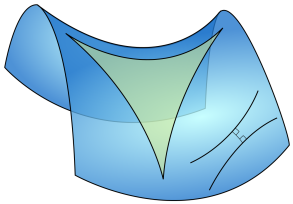
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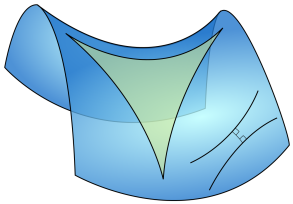


- A triangle in hyperbolic space has angle sum less than 180°.

Hyperbolic space

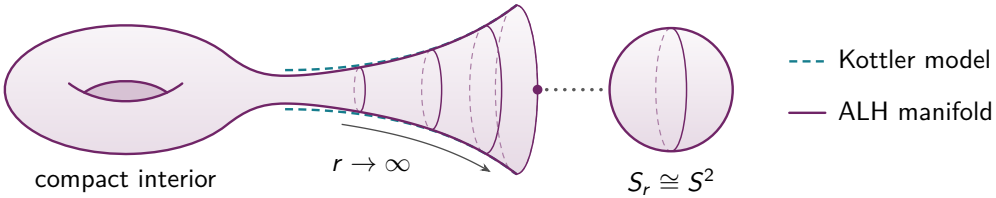
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- A triangle in hyperbolic space has angle sum less than 180°.
- The two ultraparallel lines never meet. They spread farther apart in both directions from their shortest connecting segment.

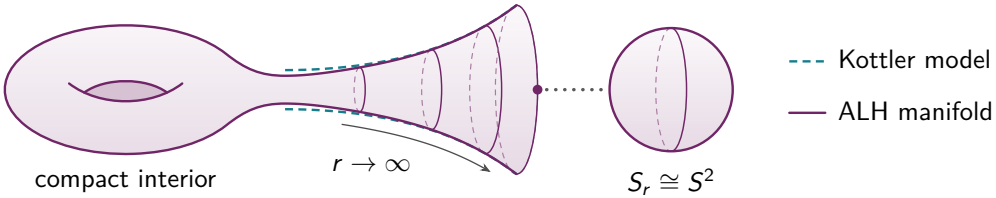
ALH manifolds



Definition (ALH manifold)

An asymptotically locally hyperbolic (ALH) manifold is a Riemannian manifold whose metric approaches a Kottler reference metric at a specified rate r^{-q} toward infinity.

ALH manifolds

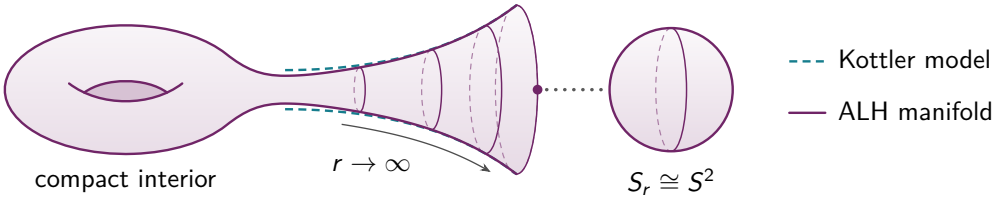


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- The interior may contain complicated geometry.
- Cross-sections of the end can be spheres, tori, higher-genus surfaces, and more.

Ricci Curvature

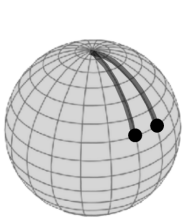
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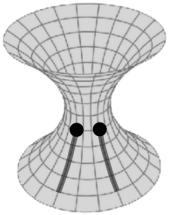
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Sperical (>0)



Euclidean (=0)



Hyperbolic (<0)

Ricci Flow

Definition (Normalized Ricci flow)

Normalized Ricci flow is an evolution of the metric $g(t)$ satisfying

$$\frac{\partial g}{\partial t} = -2(\mathrm{Ric}(g) + (n-1)g).$$

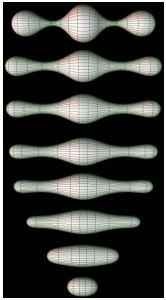
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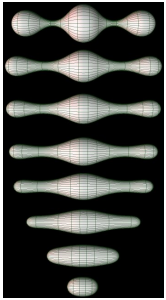
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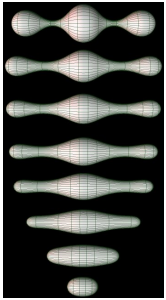
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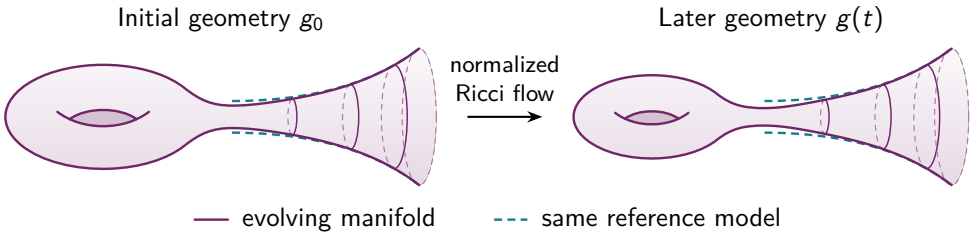
Normalized Ricci flow is an evolution of the metric $g(t)$ satisfying

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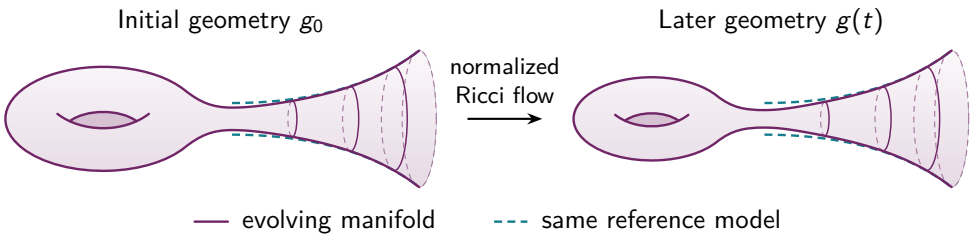
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- Perelman used Hamilton's Ricci flow to prove the Poincaré conjecture (2002–2003).



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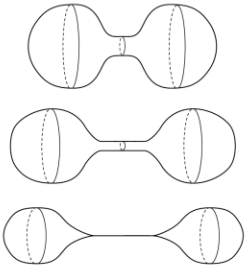


Theorem 1

Let (M^n, g_0) be a complete ALH manifold, $n \geq 3$, and let $g(t)$ evolve by normalized Ricci flow.

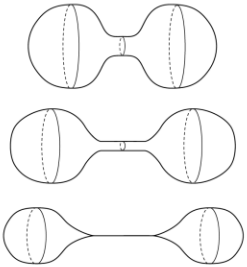
On every finite time interval where curvature stays bounded, $g(t)$ remains ALH with the same reference model at infinity.

Singularities



- A finite-time singularity forms when the Riemann curvature becomes arbitrarily large.

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- The smooth Ricci flow cannot continue through that time.

Solitons

Definition (Ricci soliton)

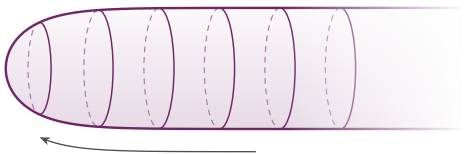
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Hamilton's cigar

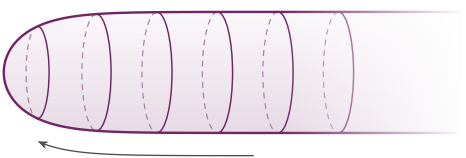


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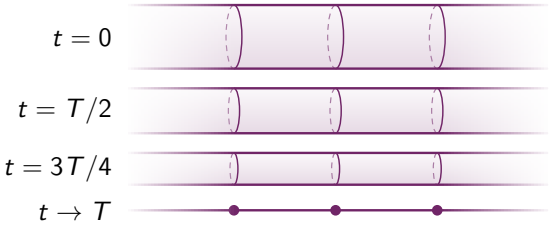
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Hamilton's cigar



Round cylinder (neckpinch model)



Result 2: Soliton rigidity

Theorem 2

Every expanding Ricci soliton in our finite-end ALH class, with the normalization used here, is Einstein:

$$\mathrm{Ric}(g) = -(n-1)g.$$

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Under normalized Ricci flow

The metric stays the same:

$$\frac{\partial g}{\partial t} = -2(\mathrm{Ric}(g) + (n-1)g) = 0, \qquad g(t) = g(0).$$

Acknowledgments

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- Everyone here for listening to my talk

References

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