On the Cardinality of the Smith Set With Many Alternatives

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Background: Voting Regime

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- Each voter provides a preference ranking over all alternatives
- Ex. Pizza $\succ$ Tacos $\succ$ Hamburgers

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Luce Model

Each alternative has a weight $\lambda_i \in [A, B]$ representing how likely it is to be ranked highly. Each voter independently samples a preference ranking from the following distribution:

$$\mathbb{P}(i_1 \succ i_2 \succ ... \succ i_n) = \frac{\lambda_{i_1}}{\lambda_{i_1} + ... + \lambda_{i_n}} \times \frac{\lambda_{i_2}}{\lambda_{i_2} + ... + \lambda_{i_n}} \times ... \times \frac{\lambda_{i_{n-1}}}{\lambda_{i_{n-1}} + \lambda_{i_n}}.$$

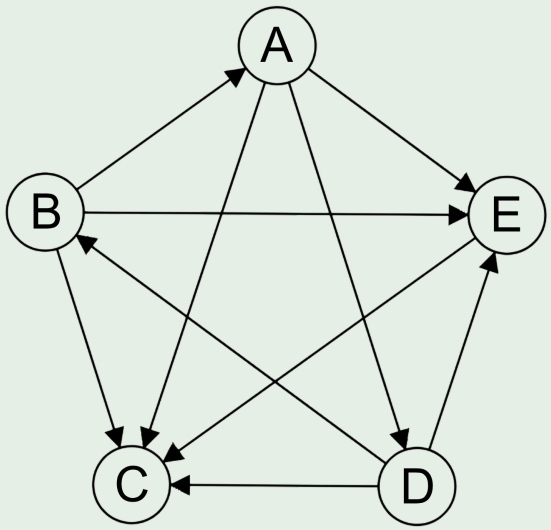
Background: Majority Digraph

Strict Majority Digraph

A graph $G = (V, E)$ where V is the set of alternatives and $(v_1, v_2) \in E$ if a strict majority of voters prefer v_1 over v_2.

Example

$$A \succ B \succ C \succ D \succ E$$
$$D \succ B \succ A \succ E \succ C$$
$$E \succ B \succ A \succ D \succ C$$



Background: Condorcet Winner

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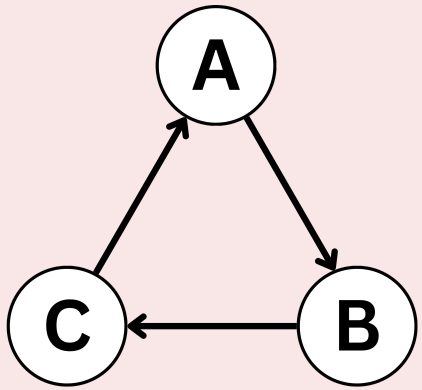
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Condorcet Paradox

Consider three alternatives A, B, C, and three voters who rank

$$A \succ B \succ C \qquad B \succ C \succ A$$

$$C \succ A \succ B$$



Then, no Condorcet winner exists.

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Smith Set

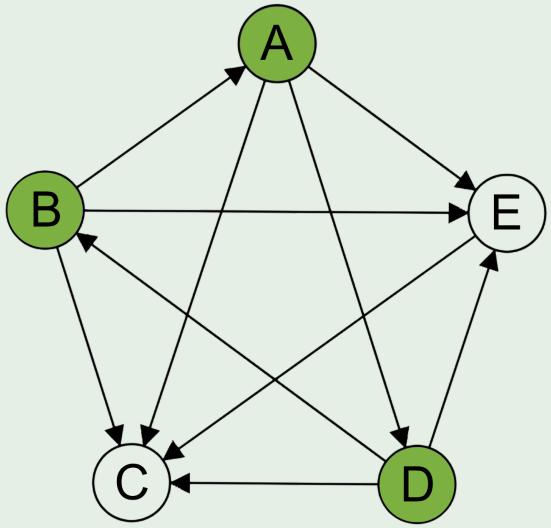
The Smith set S is the smallest dominating set.

Background: Smith Set + Majority Digraph

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$$S = \{A, B, D\}$$

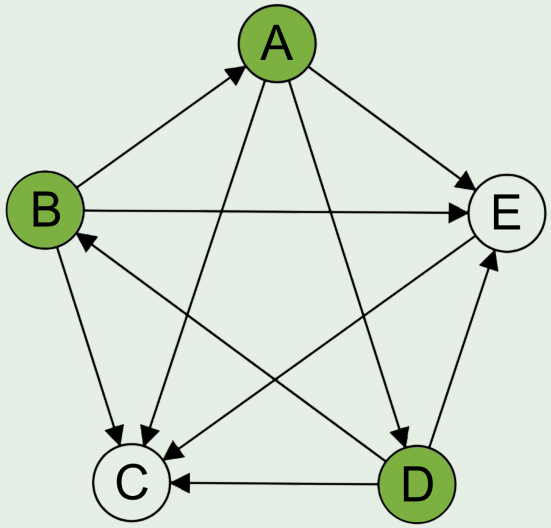


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Majority Cycle

A set of alternatives forms a majority cycle if their induced subgraph is Hamiltonian.

Our Question

Let $S_{m,n}$ be the Smith set under a specified voting regime with an odd number of voters m and $n \to \infty$ alternatives. For any integer $1 \leq s \leq n$, what is the asymptotic probability that $|S_{m,n}| = s$?

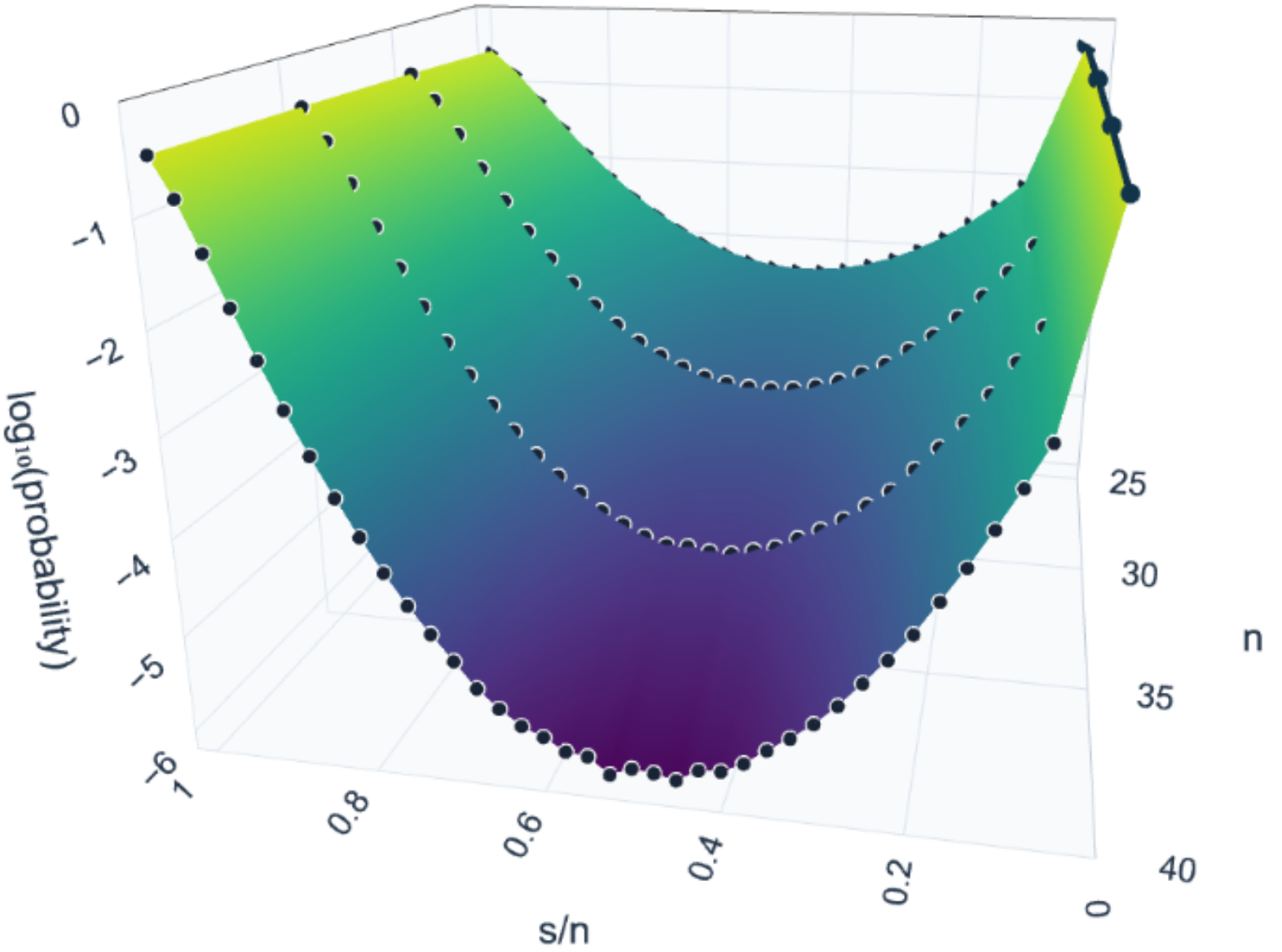
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We conjectured that majority cycles occur with high probability, so we used the upper bound

$$\mathbb{P}(S_{m,n} = S) \leq \mathbb{P}(S \text{ is dominating}).$$

Simulations



Impartial Culture and Luce Model Results

Theorem

For all $1 \leq s \leq n-1$

$\min\{s, n-s\}$	$\mathbb{P}(\lvert S_{m,n}\rvert = s)$ *Decay Rate*
Constant	*Polynomial*
$\omega(1)$	*Super-Polynomial*
$\Theta(n)$	*Exponential*

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Theorem (Majority Cycle)

The probability that the majority digraph has a Hamiltonian cycle is

$$1 - O\left(n^{-(\ell-1)/\ell}\right).$$

Impartial Culture Results

Let $m = 2\ell - 1$. Then,

Theorem

Let γ_m be a constant whose formula is omitted for conciseness. Then, all integers $1 \leq s \leq n/2$ satisfy,

$$\mathbb{P}(|S_{m,n}| = s),\ \mathbb{P}(|S_{m,n}| = n - s) \leq \left(\gamma_m \cdot \frac{s}{n}\right)^{s(\ell-1)/\ell}.$$

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Theorem

Let $\zeta(\alpha) : [0,1] \to [0,1]$ be a function whose formula is omitted for conciseness. Then, for sufficiently large n and some $\epsilon = O\left(n^{-1/(m+1)}\right)$,

$$\mathbb{P}(|S_{m,n}| = s) \leq \binom{n}{s}\left(\zeta\left(\frac{s}{n}\right) + \epsilon\right)^n.$$

Impartial Culture Results cont.

The majority cycle theorem lets us calculate tight lower bounds,

Theorem

For any fixed cardinality $s \geq 3$,

$$\mathbb{P}(|S_{m,n}| = s) = \Theta\left(n^{-\frac{s(\ell-1)}{\ell}}\right).$$

Theorem

As $n \to \infty$, uniformly over integers $n/\sqrt{\ln n} \leq s \leq n-1$,

$$\left(\frac{\mathbb{P}(|S_{m,n}| = s)}{\binom{n}{s}}\right)^{1/n} = \zeta\left(\frac{s}{n}\right) + O\left(n^{-1/(m+1)}\right).$$

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