Fair Duels after a False Start

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The duel

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- This is the *fairness rule*
- The duel itself is called the *greedy Galois game*



Example: $p = 0.1$

Shots so far	P(Alice won)	P(Bob won)	Next shooter
A	0.1000	0.0000	Bob

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ABB	0.1000	0.1710	Alice

Example: $p = 0.1$

Shots so far	P**(Alice won)**	P**(Bob won)**	**Next shooter**
A	0.1000	0.0000	Bob
AB	0.1000	0.0900	Bob
ABB	0.1000	0.1710	Alice
ABBA	0.1729	0.1710	Bob

⋮

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- For $0 < p \leq 0.2$, the sequence begins ABBABAAB...

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- Start with A; repeatedly append a copy with A and B swapped

$$\mathrm{A} \to \mathrm{A}\,\mathrm{B} \to \mathrm{AB}\,\mathrm{BA} \to \mathrm{ABBA}\,\mathrm{BAAB} \to \ldots$$

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- This sequence is not eventually periodic

Our question

Question

The players make finitely many mistakes, then follow the fairness rule forever. What happens to the shooting sequence as $p \to 0$?

A false start: AA

- Alice is erroneously given the second shot
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$$\mathrm{AA(B)^2(BA)^4(BAAB)^{12}(BAABABBA)^{91}(BAABABBAABBABAAB)^{\infty}}$$

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- Alice is erroneously given the second shot
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- This is eventually periodic!

The main theorem

Theorem

Suppose mistakes occur at finitely many fixed shot positions, including at least one after the first shot.

As $p \to 0$, the shooting sequence approaches an eventually periodic sequence.

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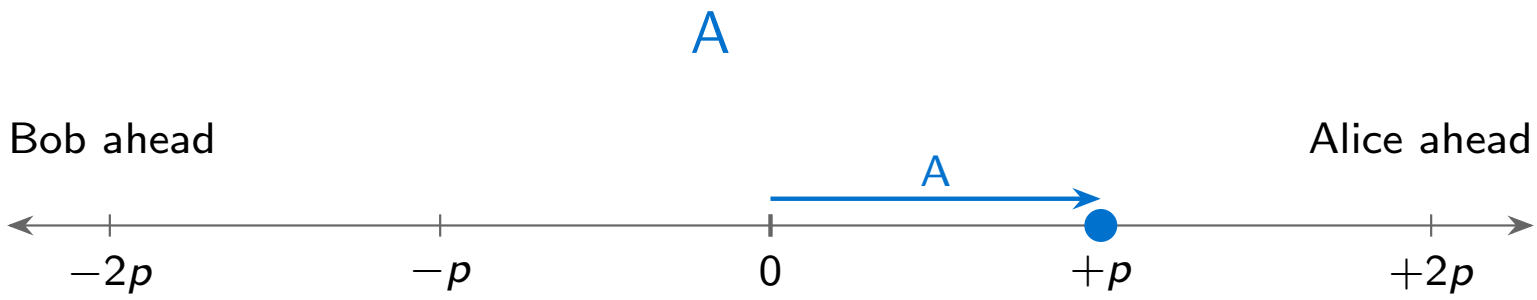
- From now on, consider very small positive p
- Advantage is $P(\text{Alice won}) - P(\text{Bob won})$
- Positive $\to$ Bob; nonpositive $\to$ Alice
- Each shot moves the advantage toward zero and may overshoot

The first four shots

- First hit on shot k: probability $p(1-p)^{k-1}$

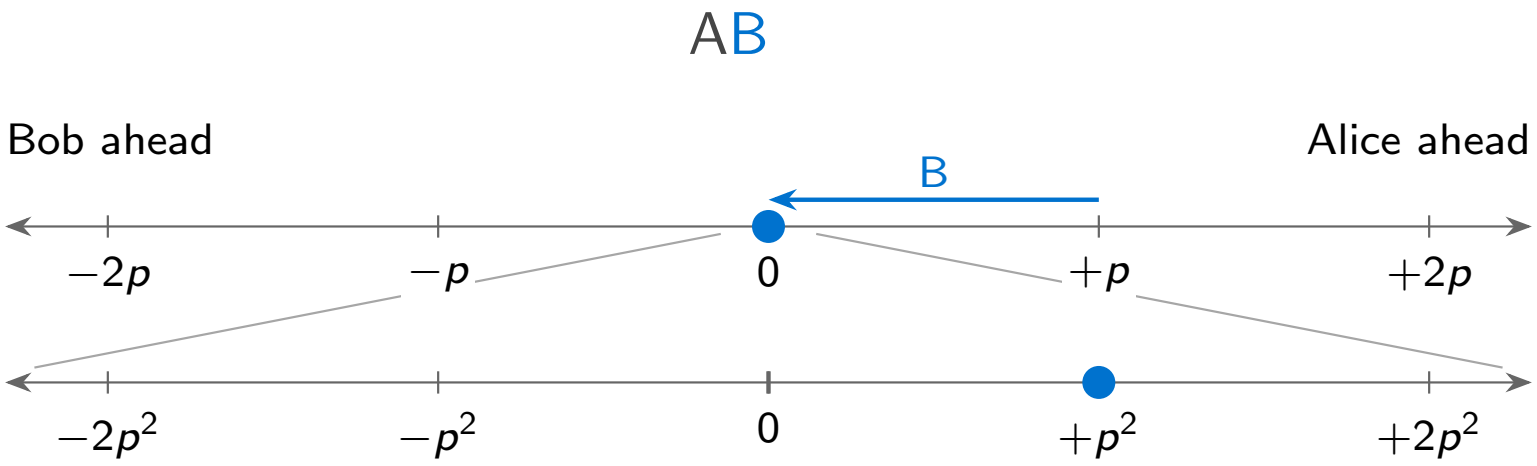
The first four shots

- First hit on shot k: probability $p(1-p)^{k-1}$
- Advantage: p; Bob shoots next
  The blue dot shows the advantage at this scale



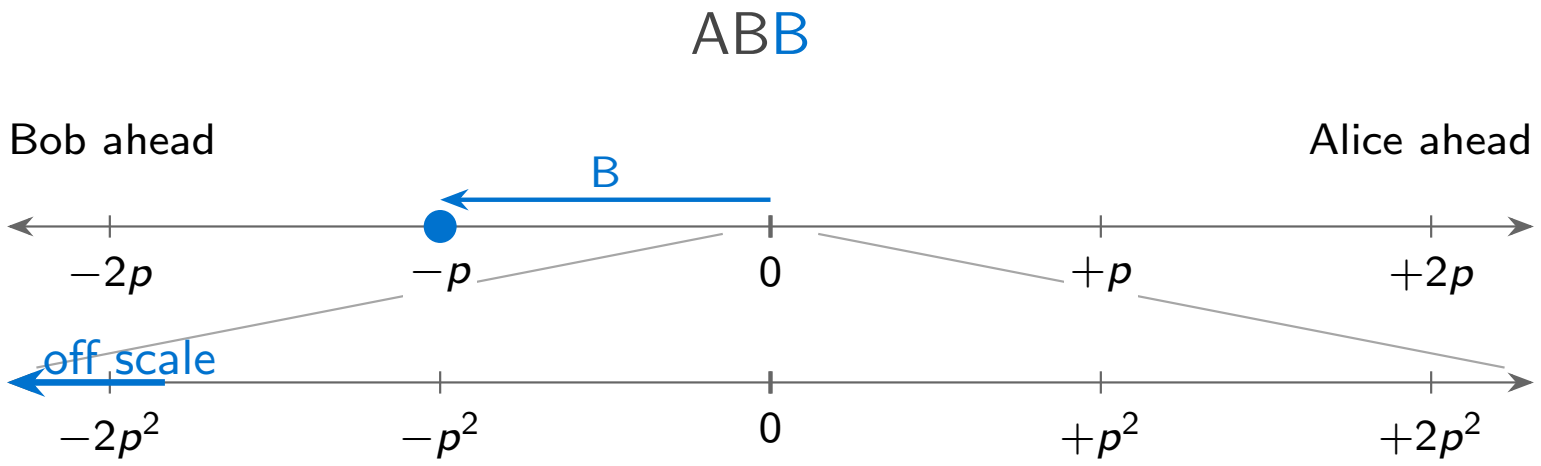
The first four shots

- First hit on shot k: probability $p(1-p)^{k-1}$
- Advantage: $p - p(1-p) = p^2$; Bob shoots next
  The p term cancels; zoom in to see the sign



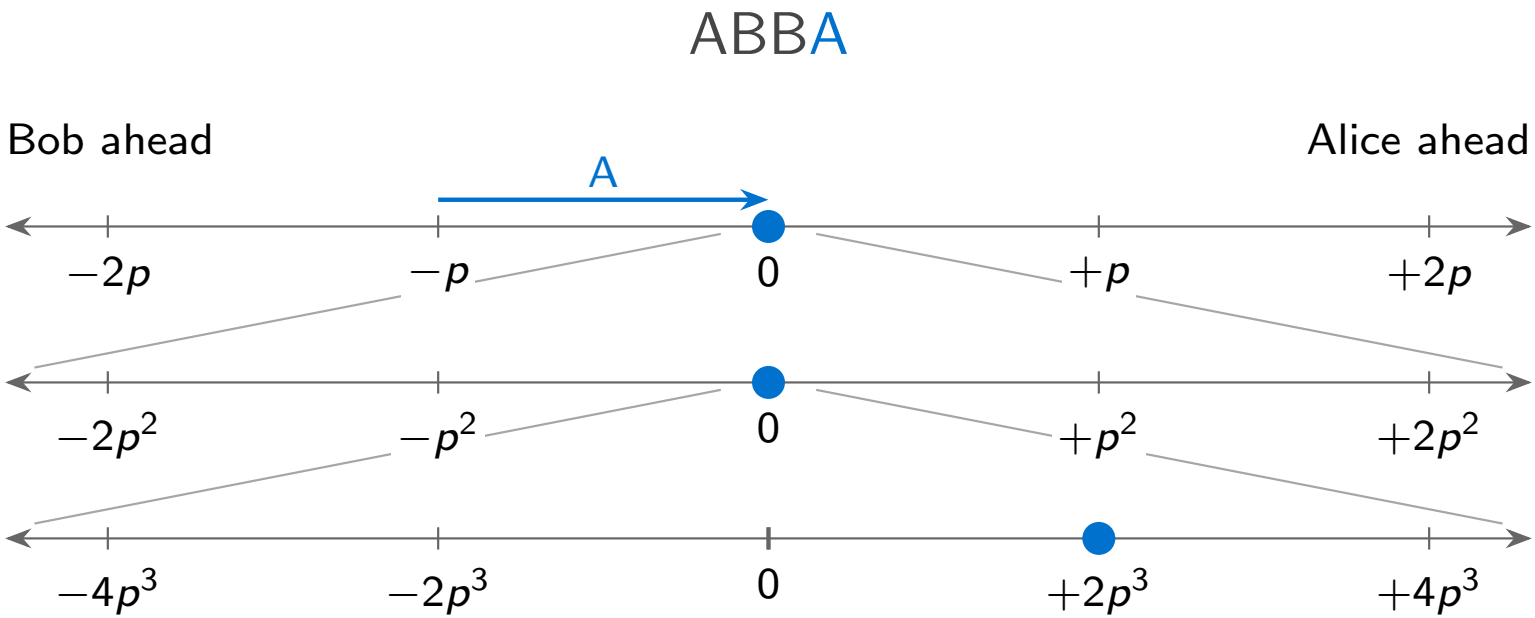
The first four shots

- First hit on shot k: probability $p(1-p)^{k-1}$
- Advantage: $p - p(1-p) - p(1-p)^2 = -p + 3p^2 - p^3$
  Alice shoots next; the leading term determines the sign



The first four shots

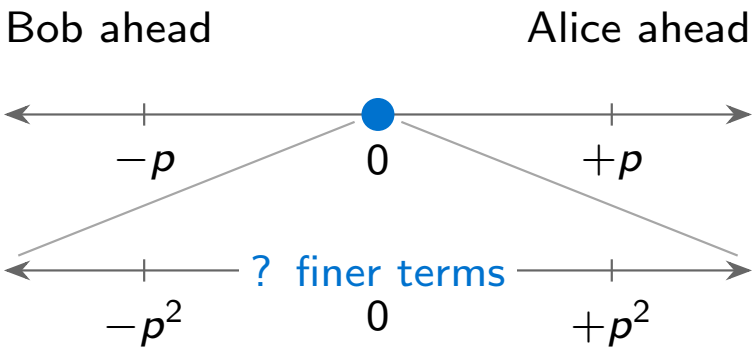
- First hit on shot k: probability $p(1-p)^{k-1}$
- Advantage: $p - p(1-p) - p(1-p)^2 + p(1-p)^3 = 2p^3 - p^4$
  Bob shoots next



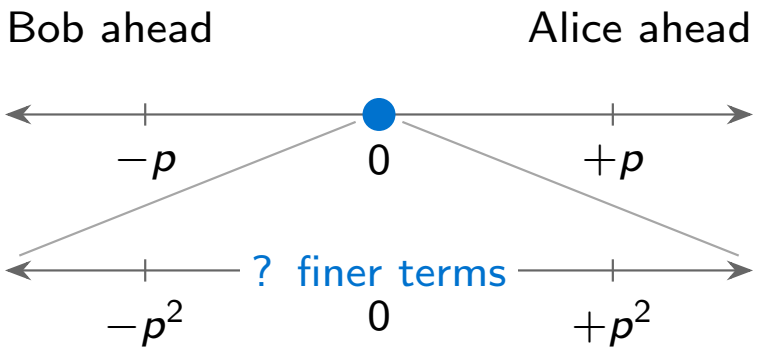
2-shot blocks

- If the advantage has no p term:

Advantage beforehand ≤ 0



Advantage beforehand > 0

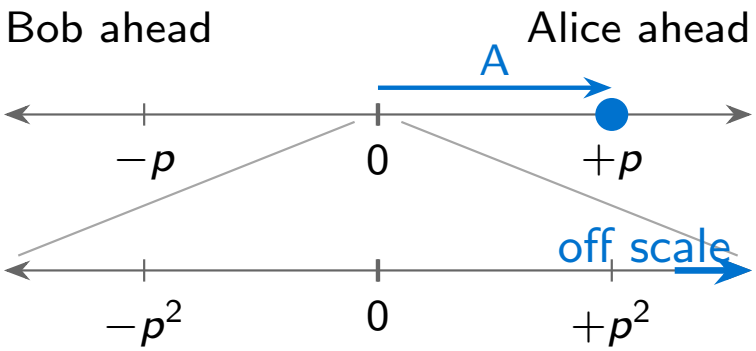


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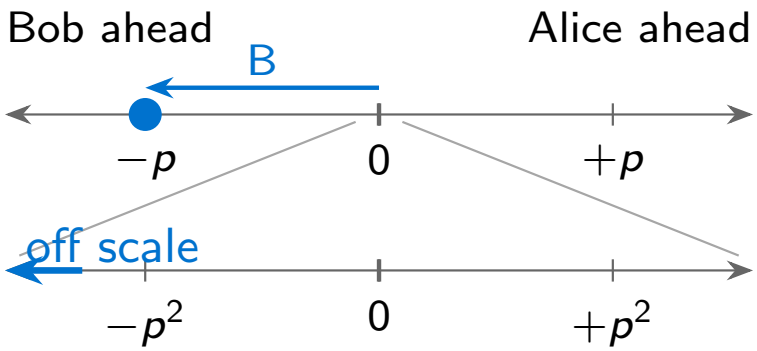
Advantage beforehand ≤ 0

A



Advantage beforehand > 0

B



2-shot blocks

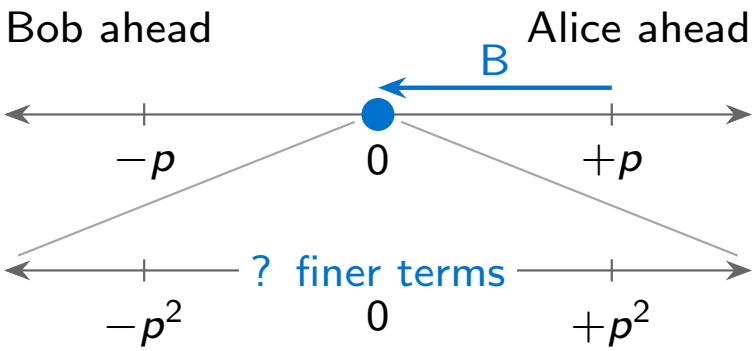
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$\underbrace{\text{A B}}$

one 2-shot block

Change in advantage: $+p^2+\cdots$

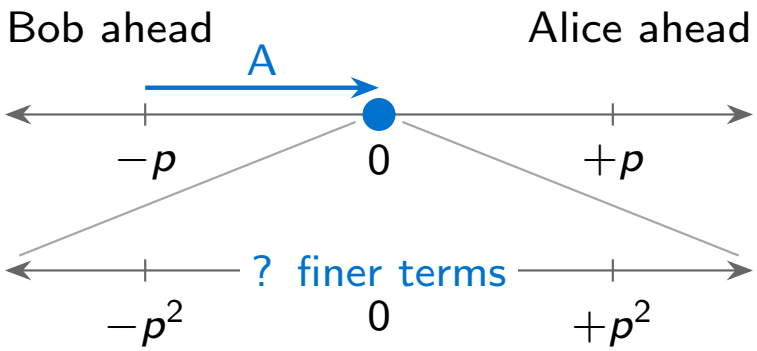


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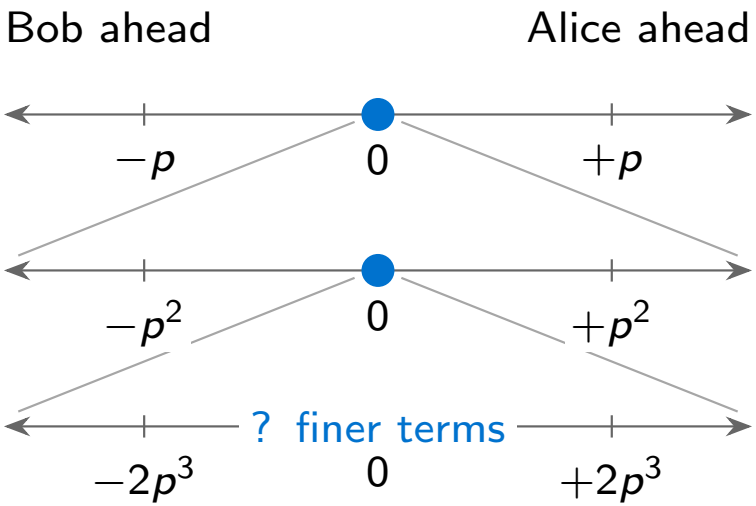
Change in advantage: $-p^2+\cdots$



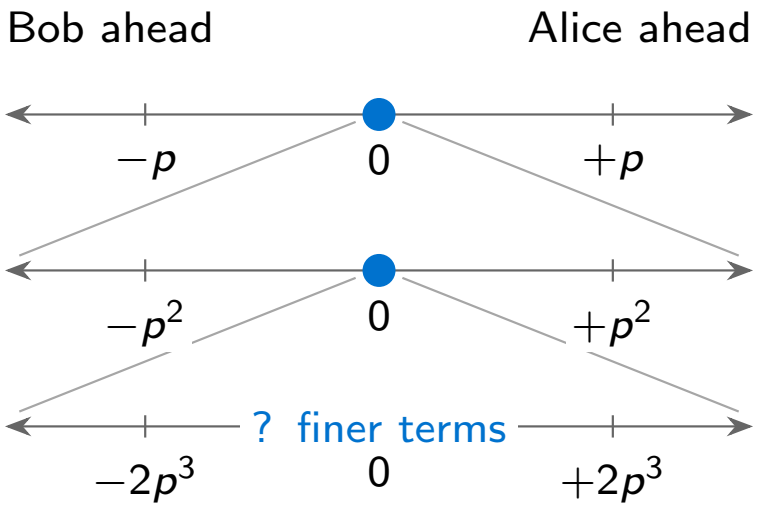
4-shot blocks

- If the advantage has no p or p^2 term:

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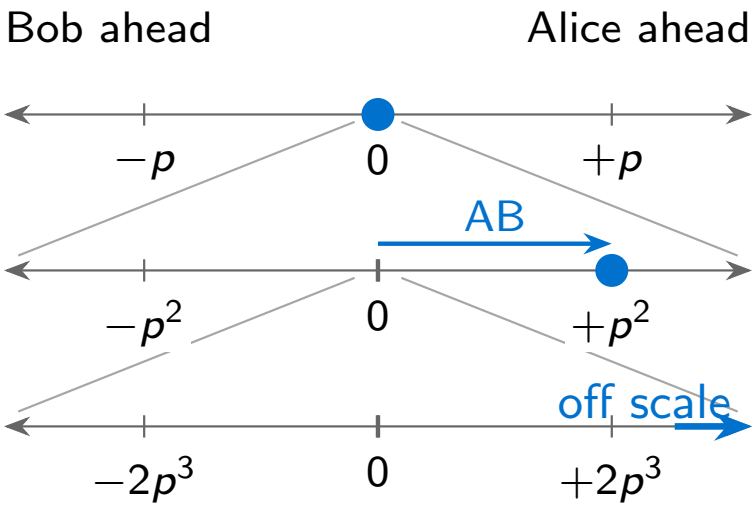


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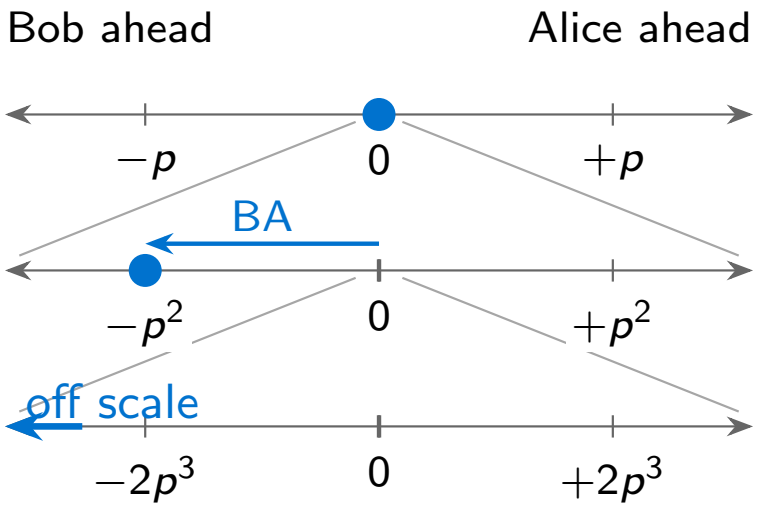
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AB



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BA



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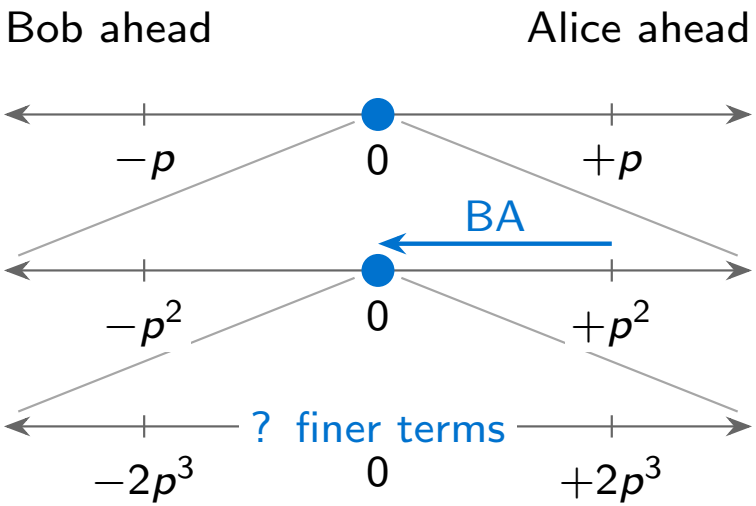
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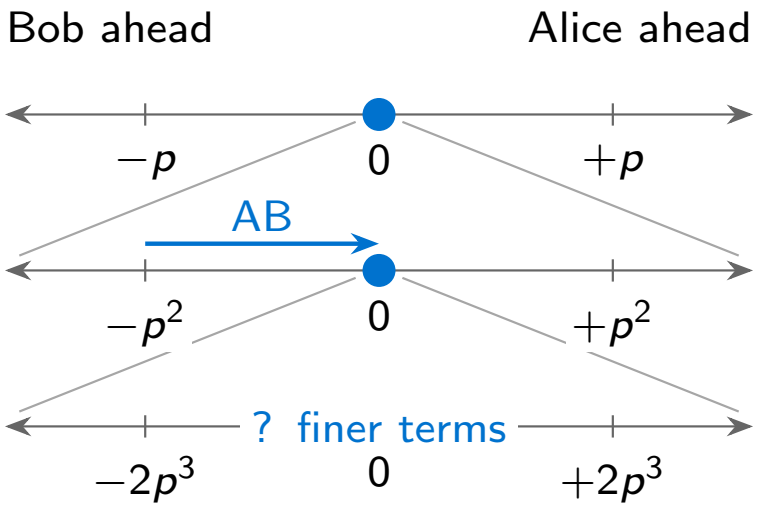


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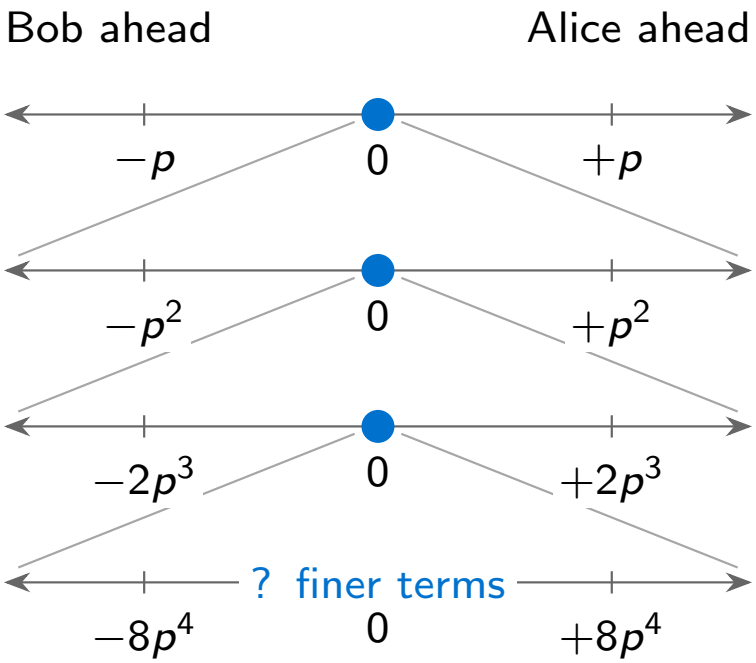
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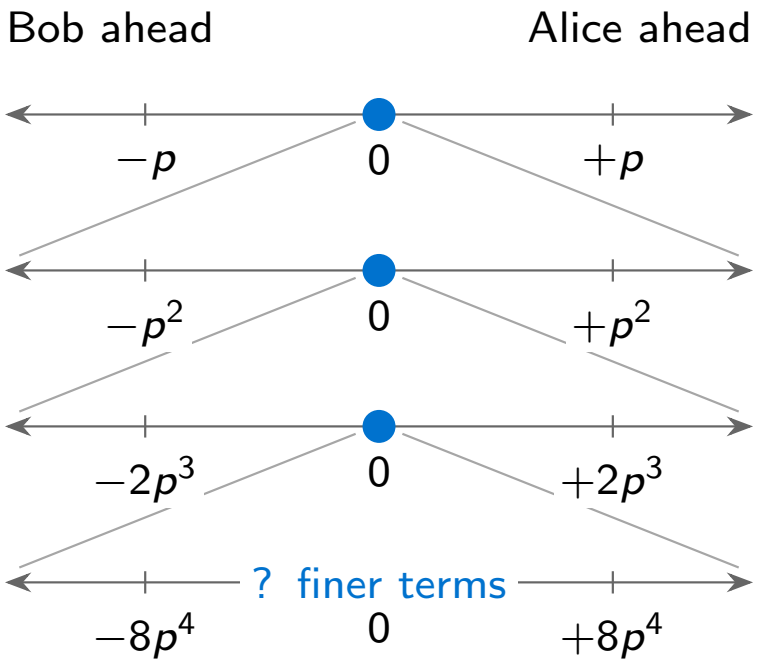
8-shot blocks

- If the advantage has no p, p^2, or p^3 term:

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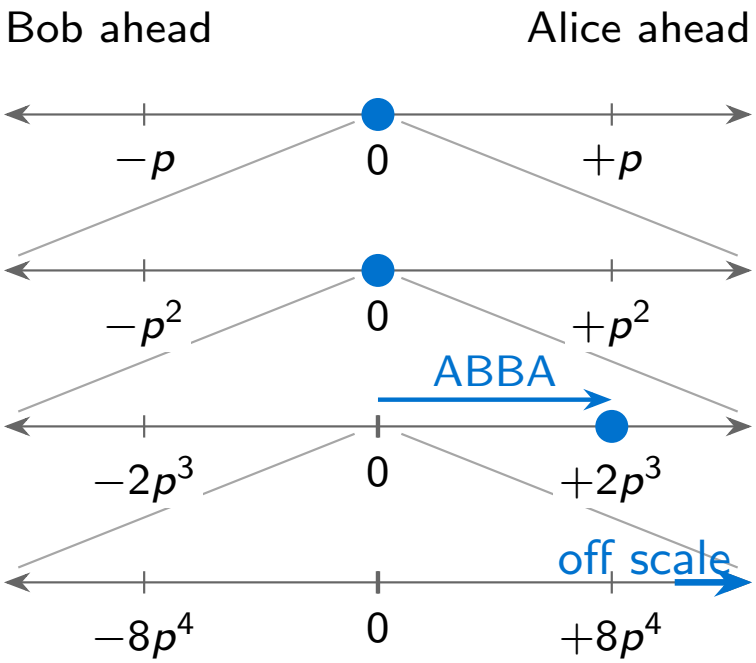


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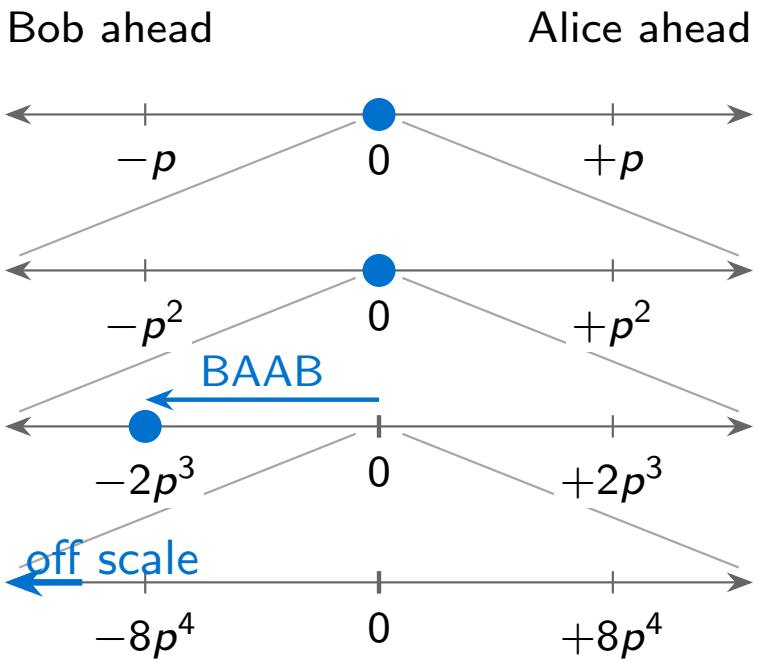
Advantage beforehand ≤ 0

ABBA



Advantage beforehand > 0

BAAB



8-shot blocks

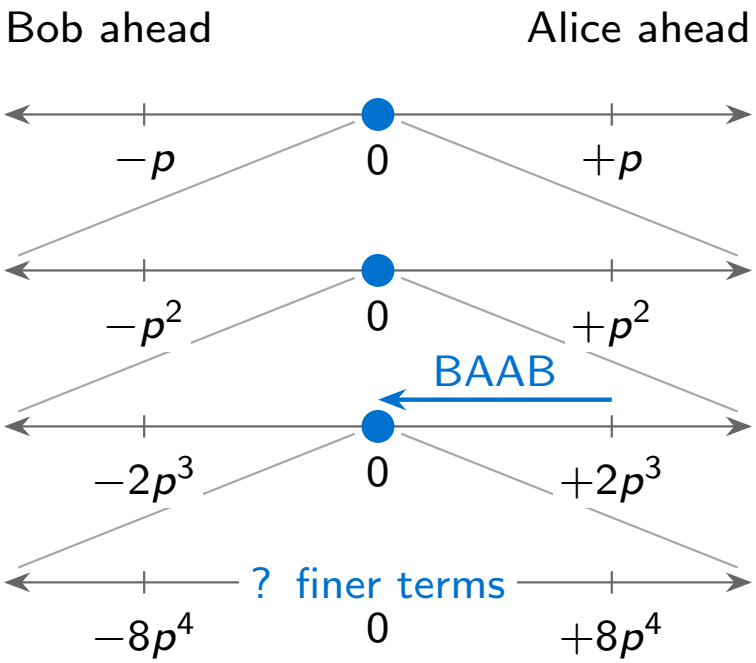
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ABBA BAAB

one 8-shot block

Change in advantage: $+8p^4 + \cdots$

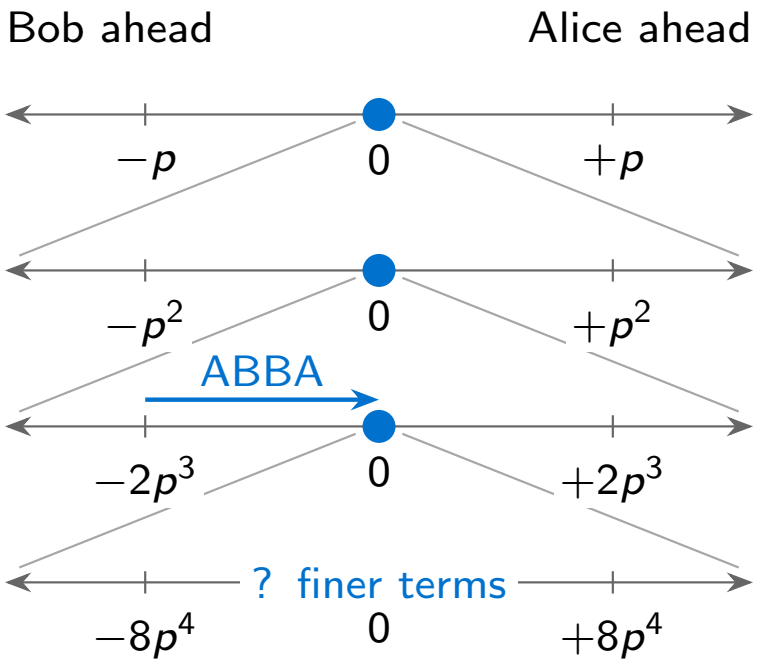


Advantage beforehand > 0

BAAB ABBA

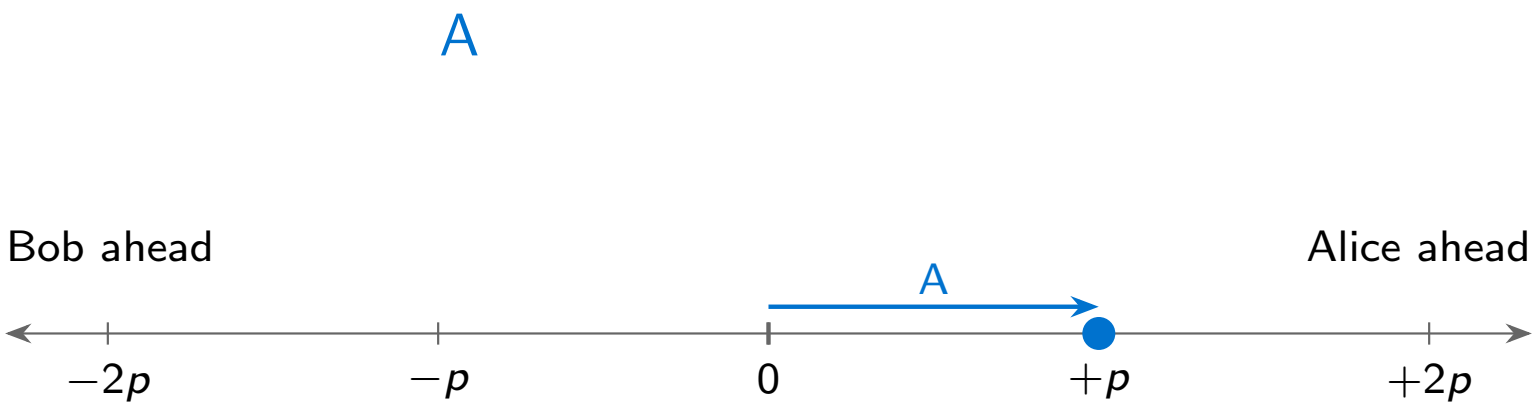
one 8-shot block

Change in advantage: $-8p^4 + \cdots$



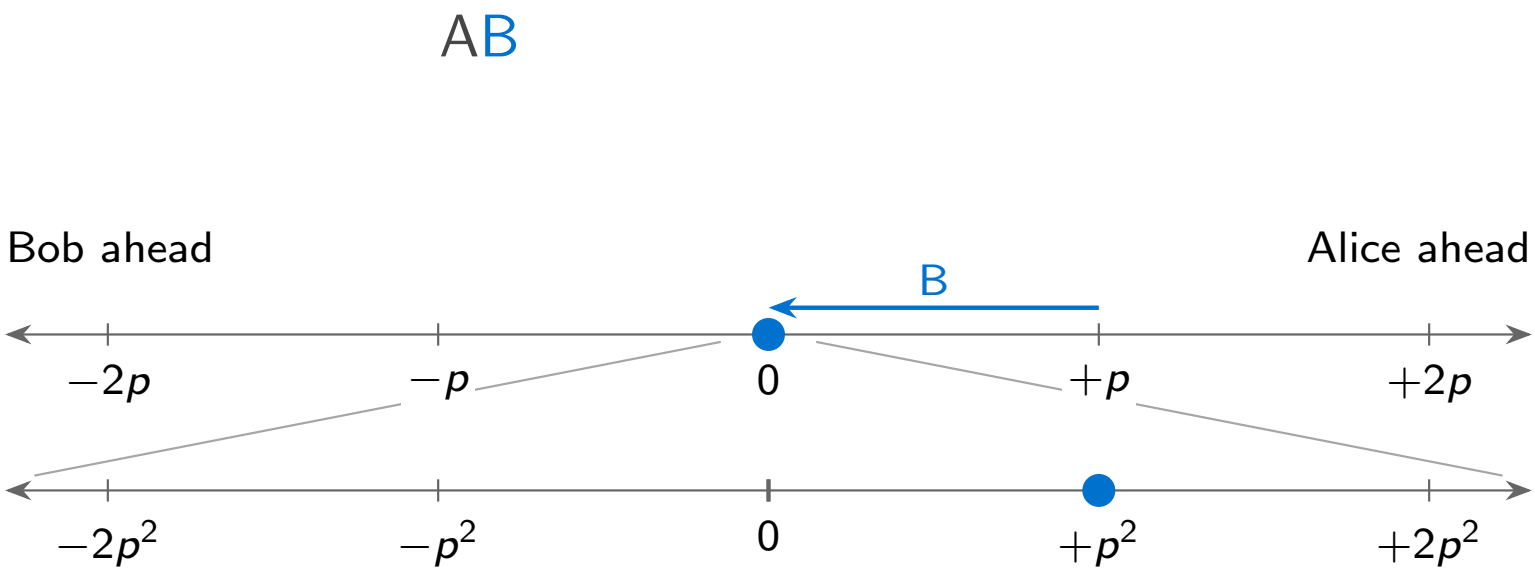
Recovering the Thue–Morse sequence

- Start with A and follow the forced blocks
  Leading advantage: p



Recovering the Thue–Morse sequence

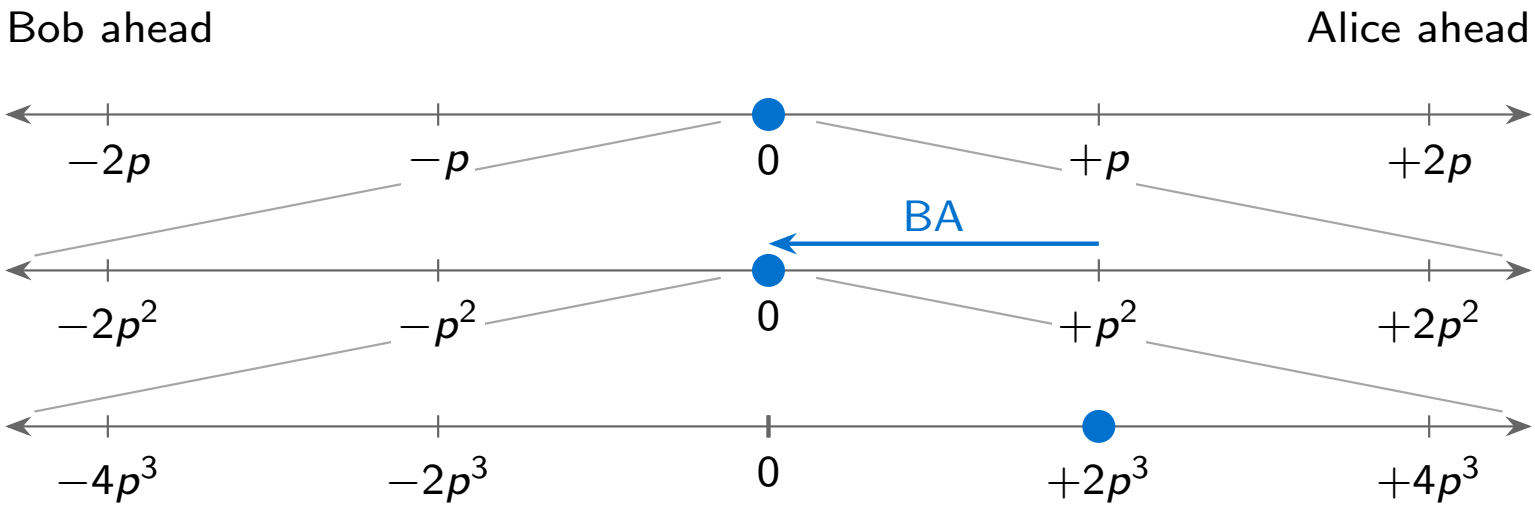
- B cancels the p term; zoom to p^2
  Leading advantage: p^2



Recovering the Thue–Morse sequence

- BA cancels the p^2 term; zoom to p^3
  Leading advantage: $2p^3$

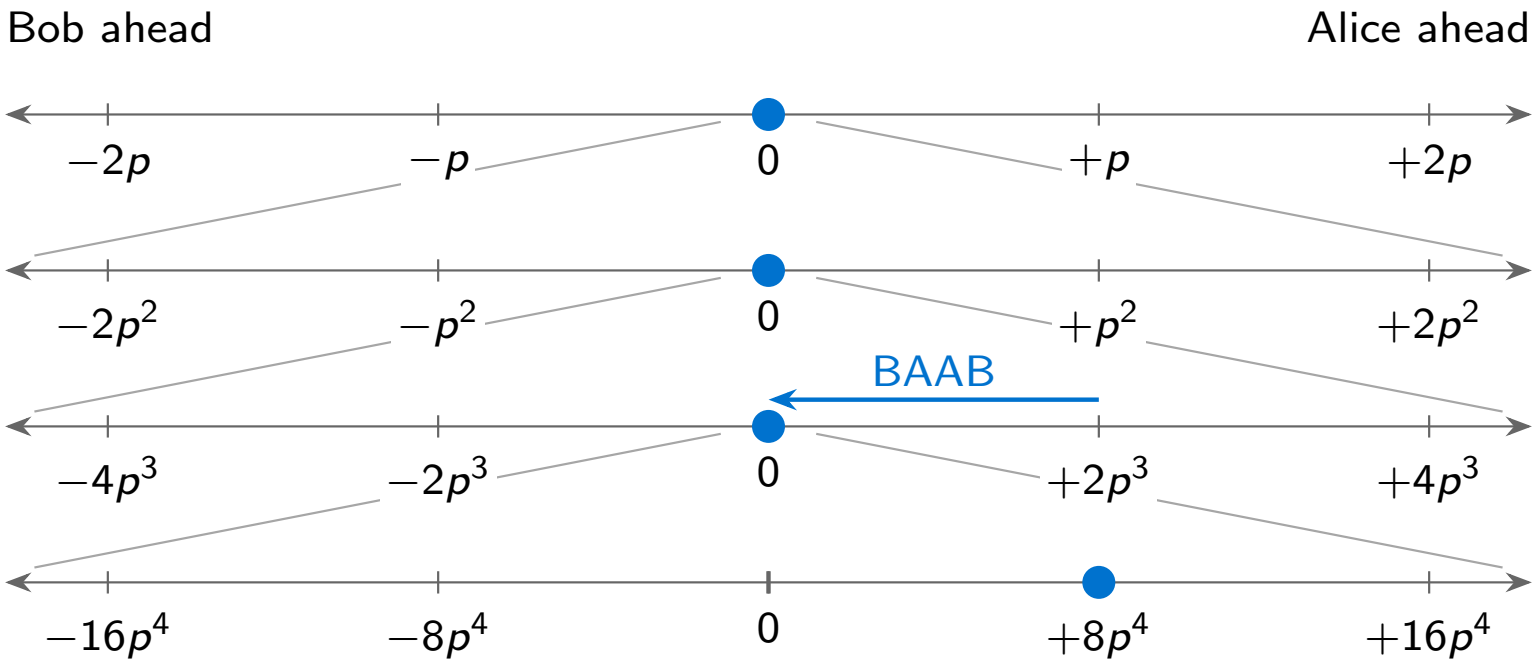
ABBA



Recovering the Thue–Morse sequence

- BAAB cancels the p^3 term; zoom to p^4
  Leading advantage: $8p^4$

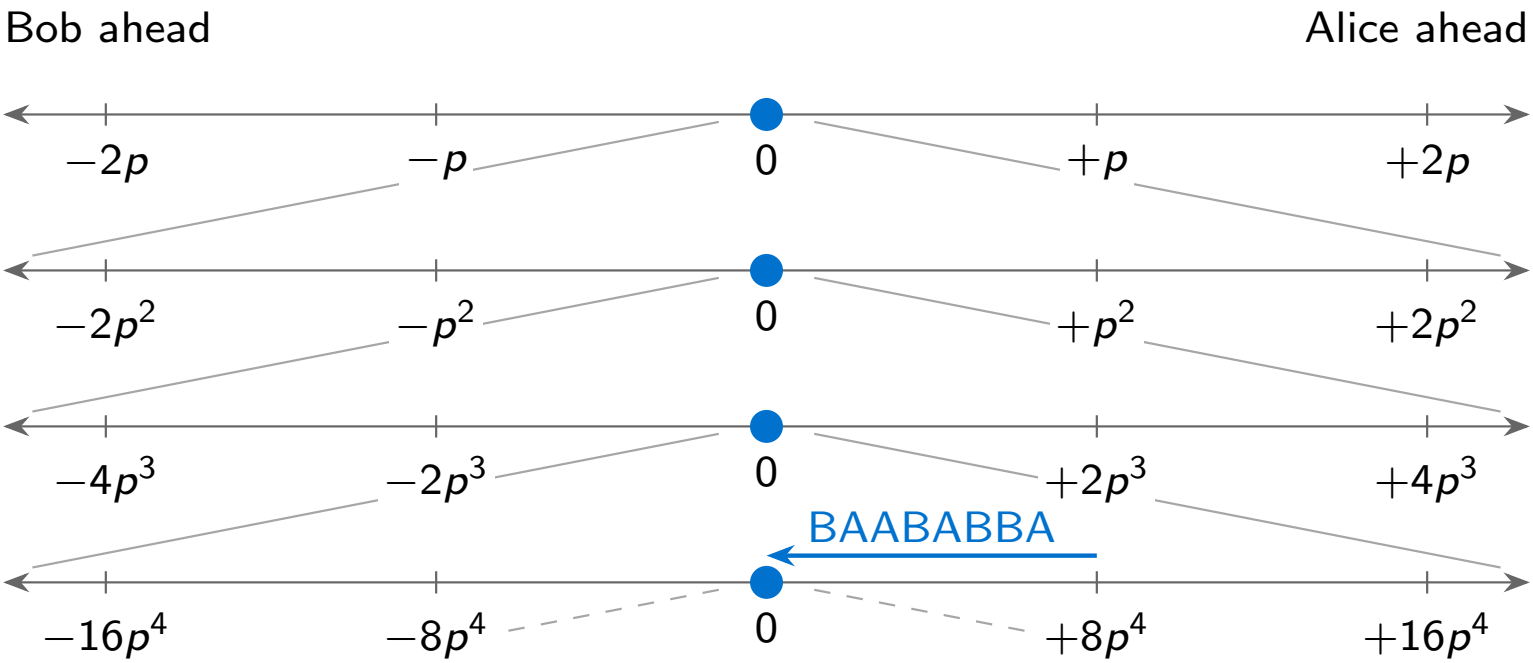
ABBABAAB



Recovering the Thue–Morse sequence

- The p^4 term cancels; continue at finer scales

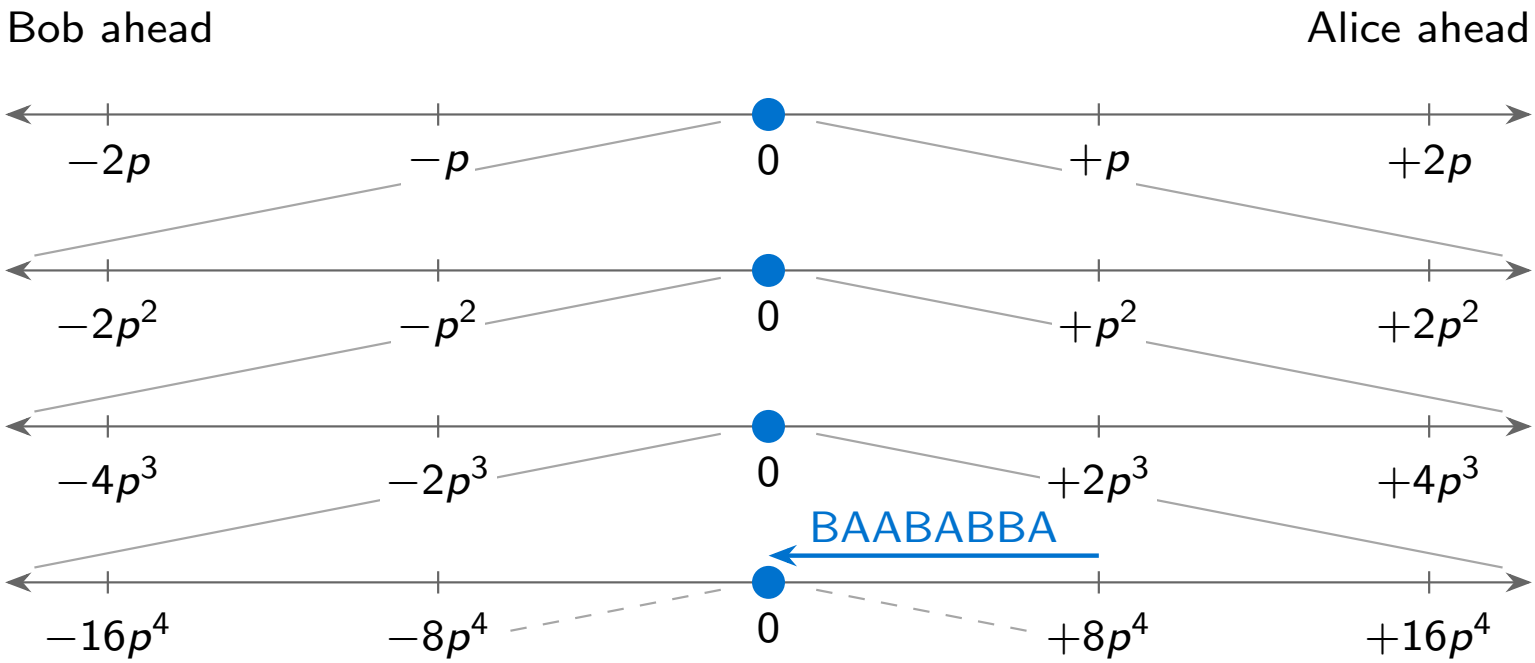
ABBABAABBAABABBA



Recovering the Thue–Morse sequence

- The p^4 term cancels; continue at finer scales
- This is the Thue–Morse construction

ABBABAABBAABABBA



Starting with AA

- Alice takes the first two shots; each shot moves the dot by p
  Leading advantage: $2p$

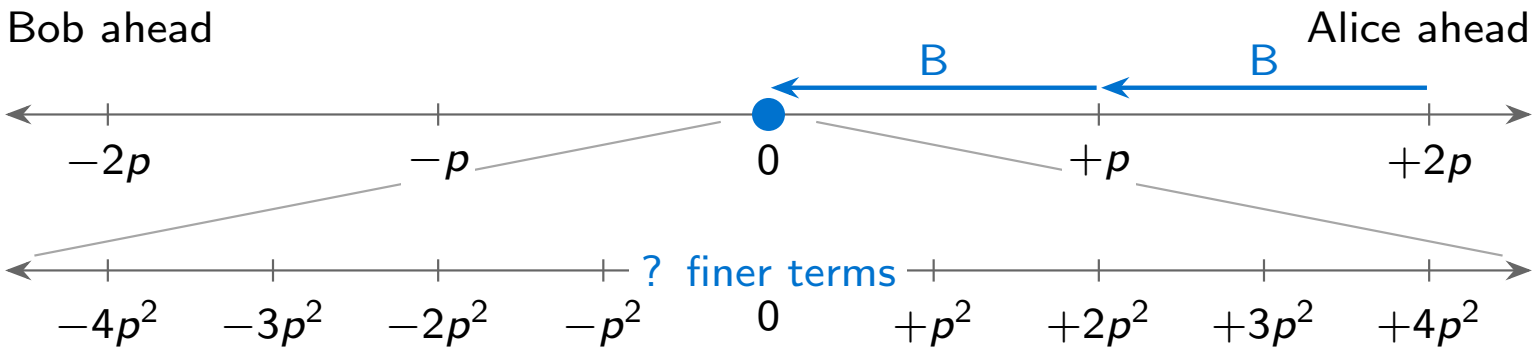
AA



Starting with AA

- 2 B shots cancel the p term; zoom to p^2
  p term: $2p - 2 \cdot p = 0$

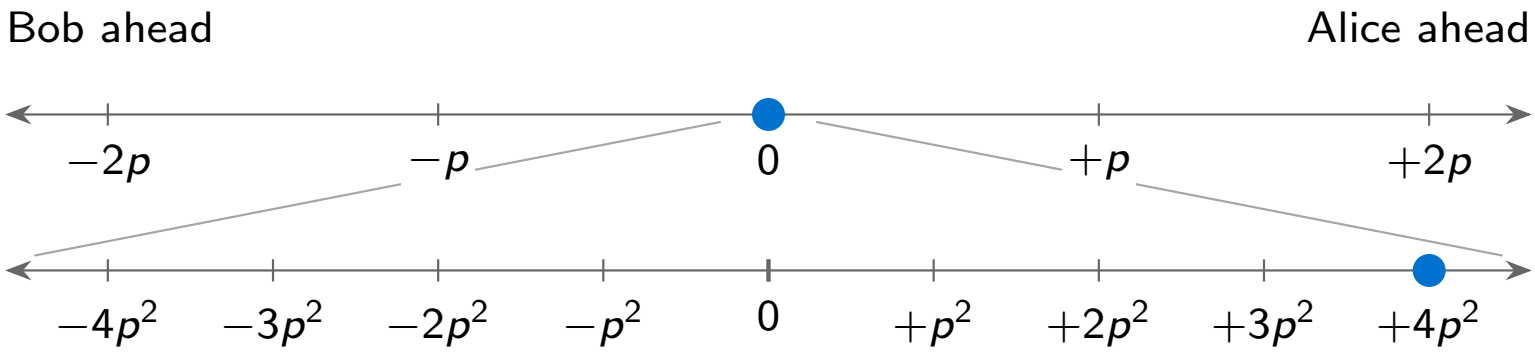
AA $(\mathrm{B})^2$



Continuing in 2-shot blocks

- The p term has canceled; 2-shot blocks move the dot by p^2
  Leading advantage: $4p^2$

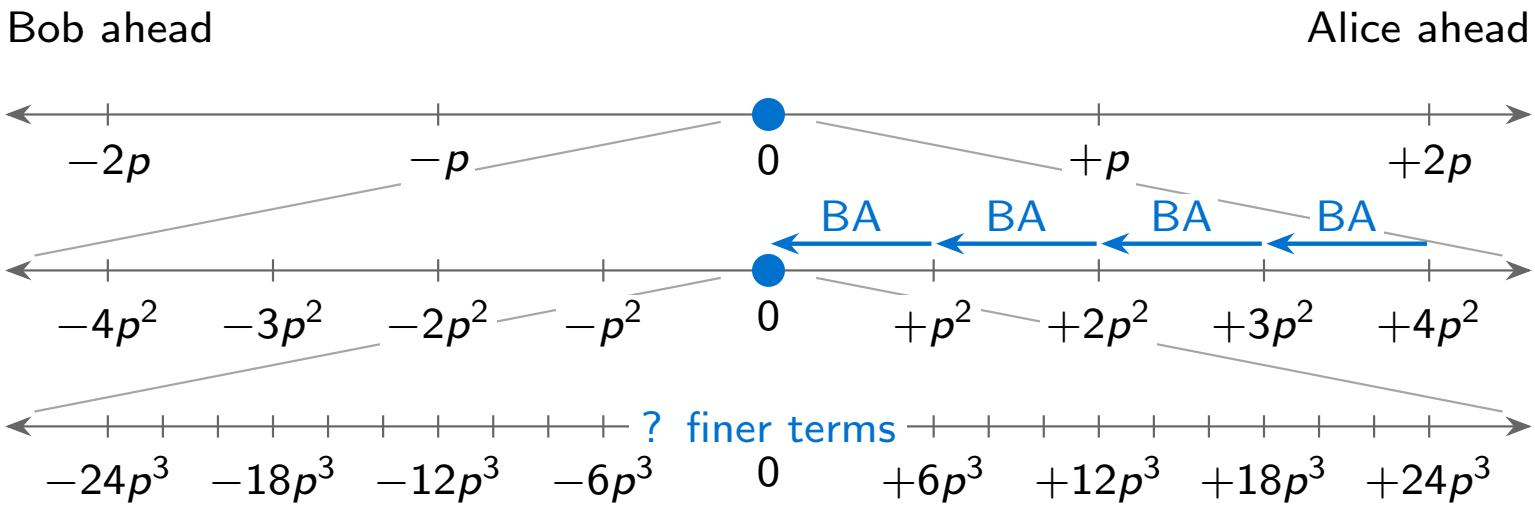
$$\mathrm{AA}\,(\mathrm{B})^2$$



Continuing in 2-shot blocks

- 4 BA blocks cancel the p^2 term; zoom to p^3
  p^2 term: $4p^2 - 4 \cdot p^2 = 0$

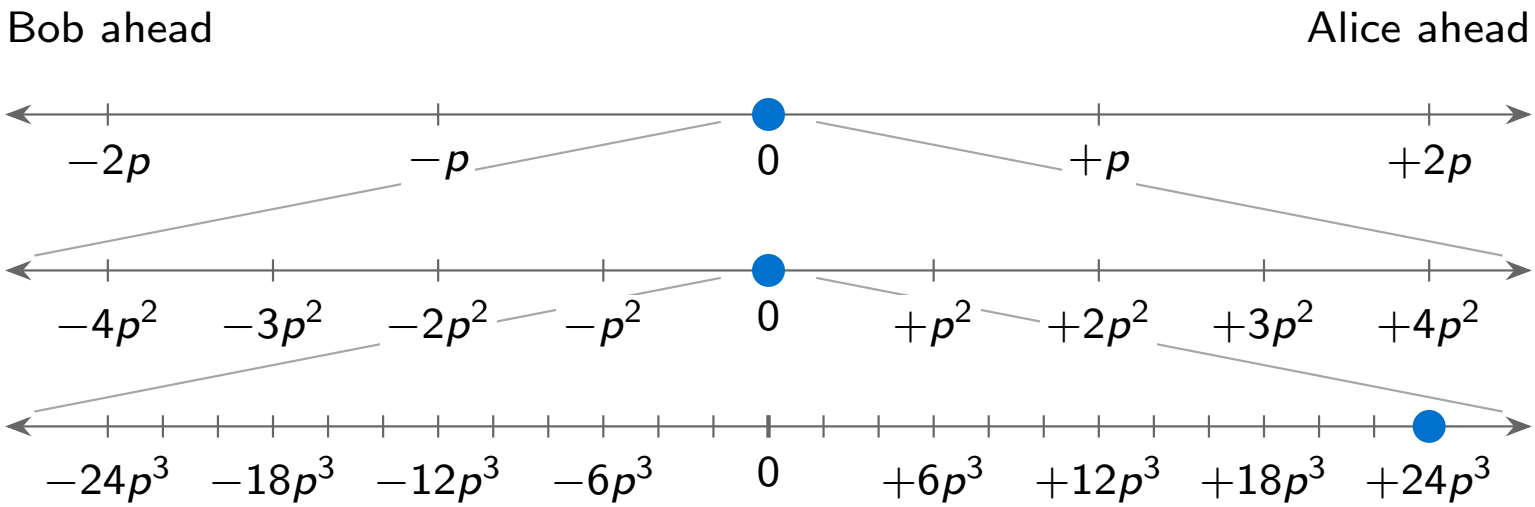
$$\mathrm{AA}\,(\mathrm{B})^2\,(\mathrm{BA})^4$$



Continuing in 4-shot blocks

- The p^2 term has canceled; 4-shot blocks move the dot by $2p^3$
  Leading advantage: $24p^3$

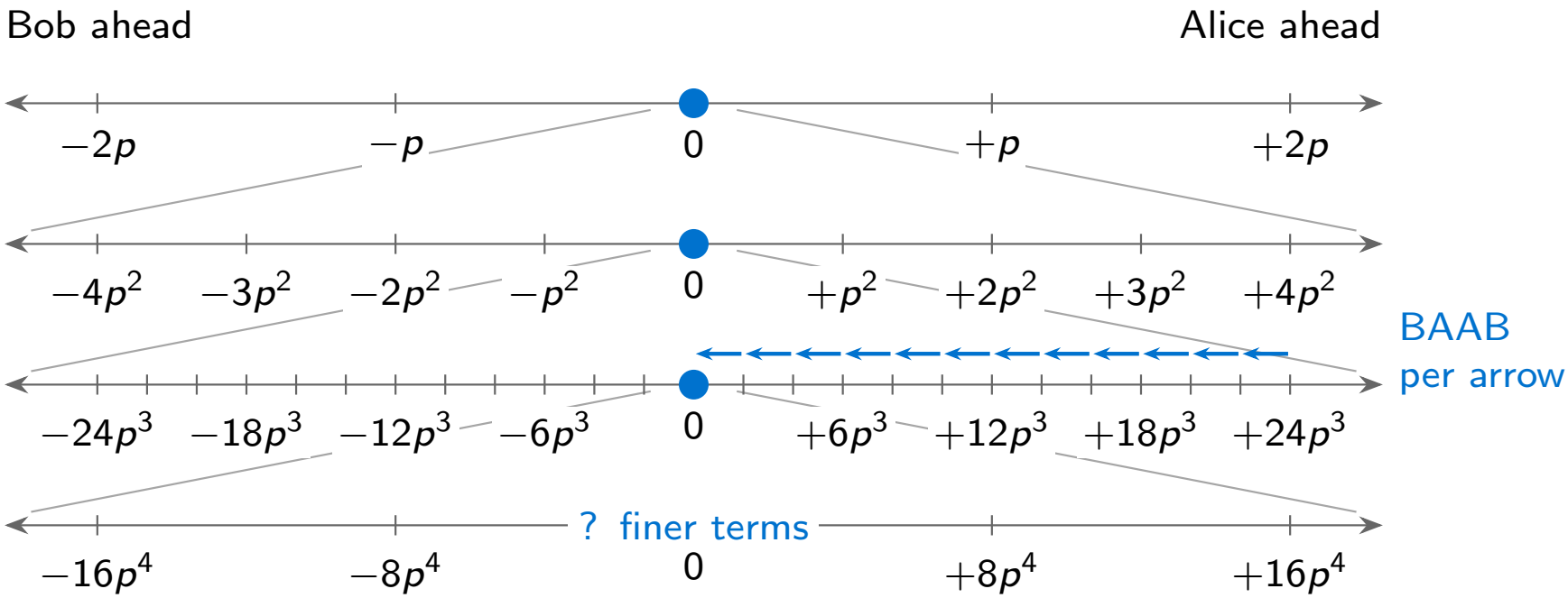
$$\mathrm{AA}\,(\mathrm{B})^2\,(\mathrm{BA})^4$$



Continuing in 4-shot blocks

- 12 BAAB blocks cancel the p^3 term; zoom to p^4
  p^3 term: $24p^3 - 12 \cdot 2p^3 = 0$

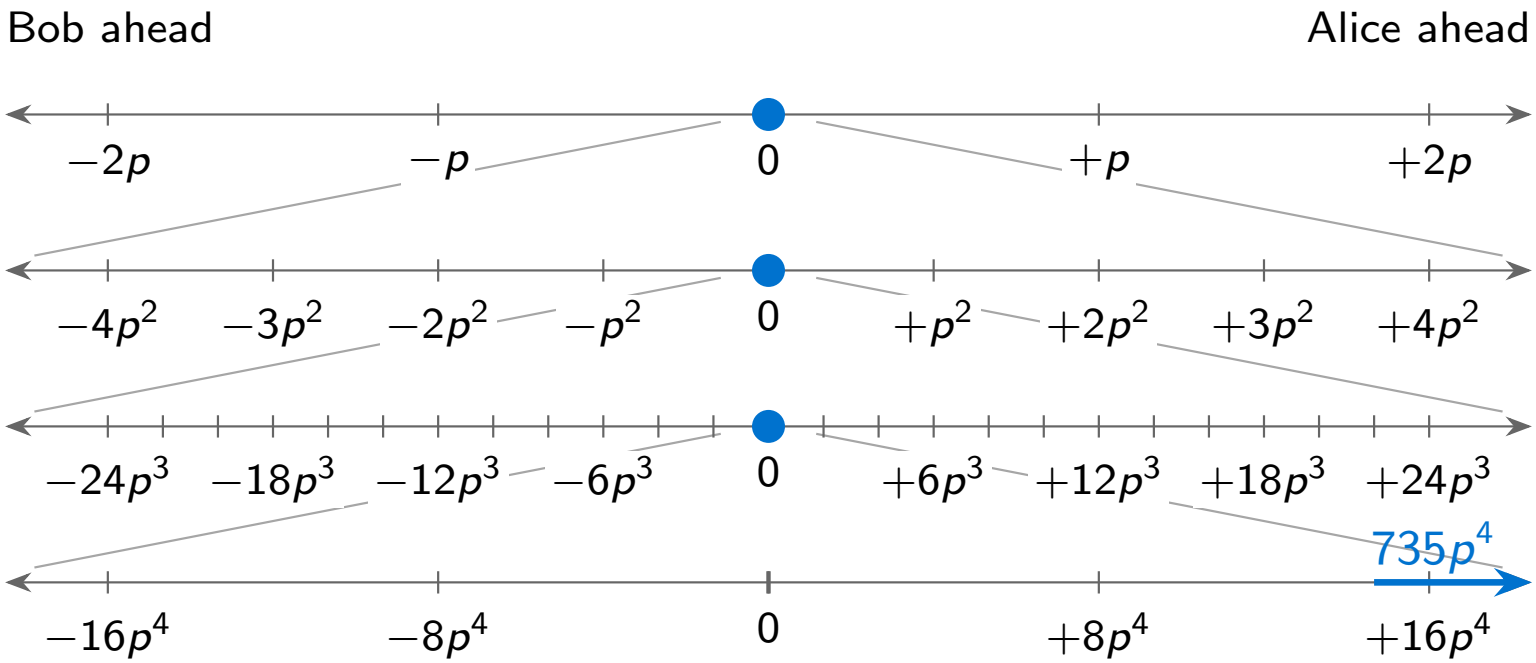
$$\mathrm{AA}\,(\mathrm{B})^2\,(\mathrm{BA})^4\,(\mathrm{BAAB})^{12}$$



Continuing in 8-shot blocks

- The p^3 term has canceled; 8-shot blocks move the dot by $8p^4$
  Leading advantage: $735p^4$

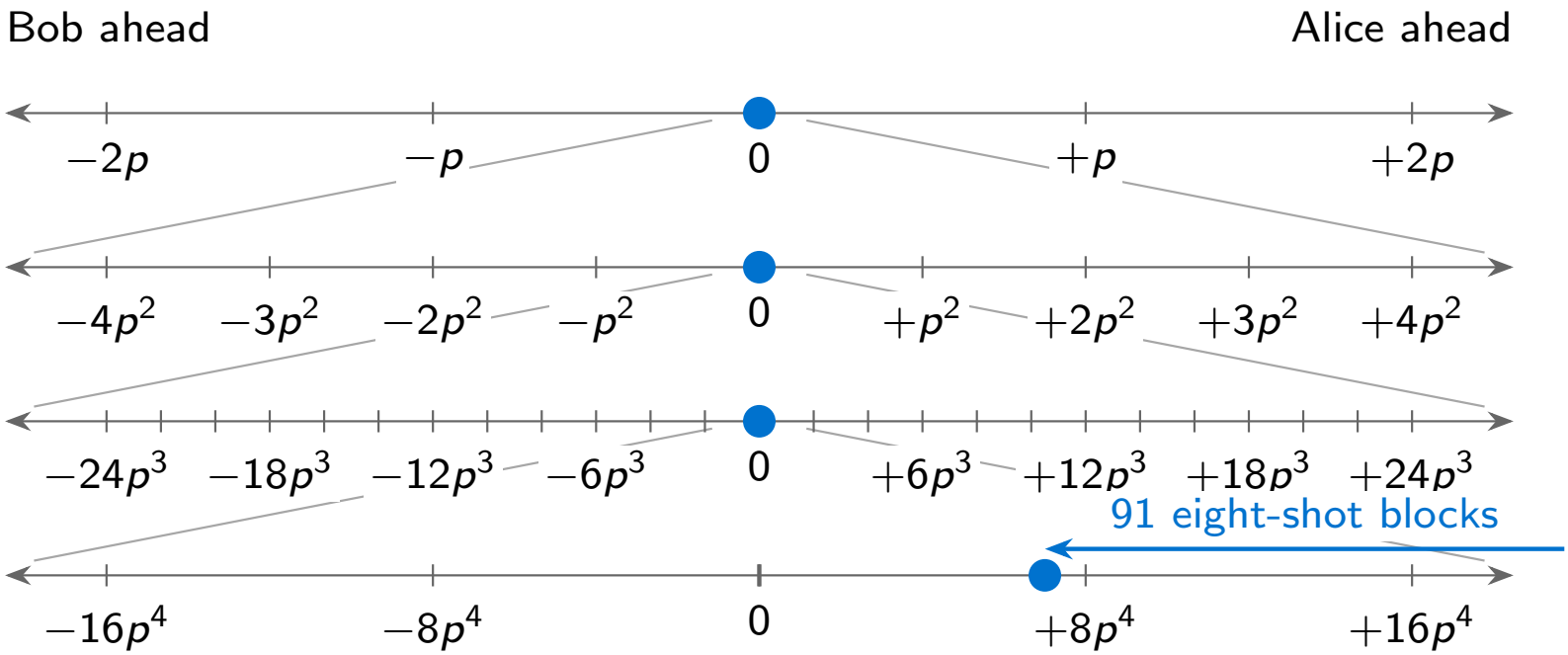
$$\mathrm{AA}\,(\mathrm{B})^2\,(\mathrm{BA})^4\,(\mathrm{BAAB})^{12}$$



Continuing in 8-shot blocks

- The p^4 term does not cancel
  p^4 term: $735p^4 - 91 \cdot 8p^4 = 7p^4$

$$\mathrm{AA}\,(\mathrm{B})^2\,(\mathrm{BA})^4\,(\mathrm{BAAB})^{12}$$
$$(\mathrm{BAABABBA})^{91}$$



The repeating tail

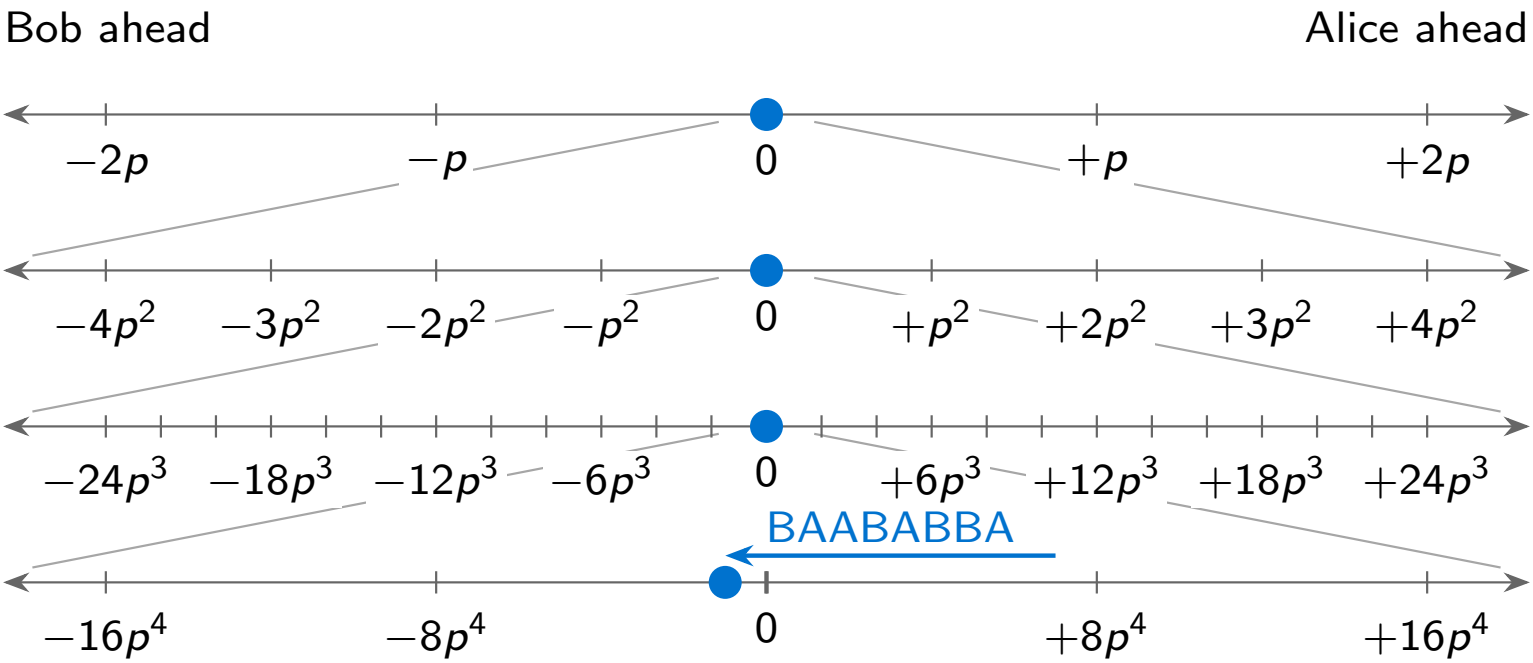
- Positive advantage forces BAABABBA

  p^4 term: $7p^4 - 8p^4 = -p^4$

  $\mathrm{AA}\,(\mathrm{B})^2\,(\mathrm{BA})^4\,(\mathrm{BAAB})^{12}$

  $(\mathrm{BAABABBA})^{91}$

  BAABABBA



The repeating tail

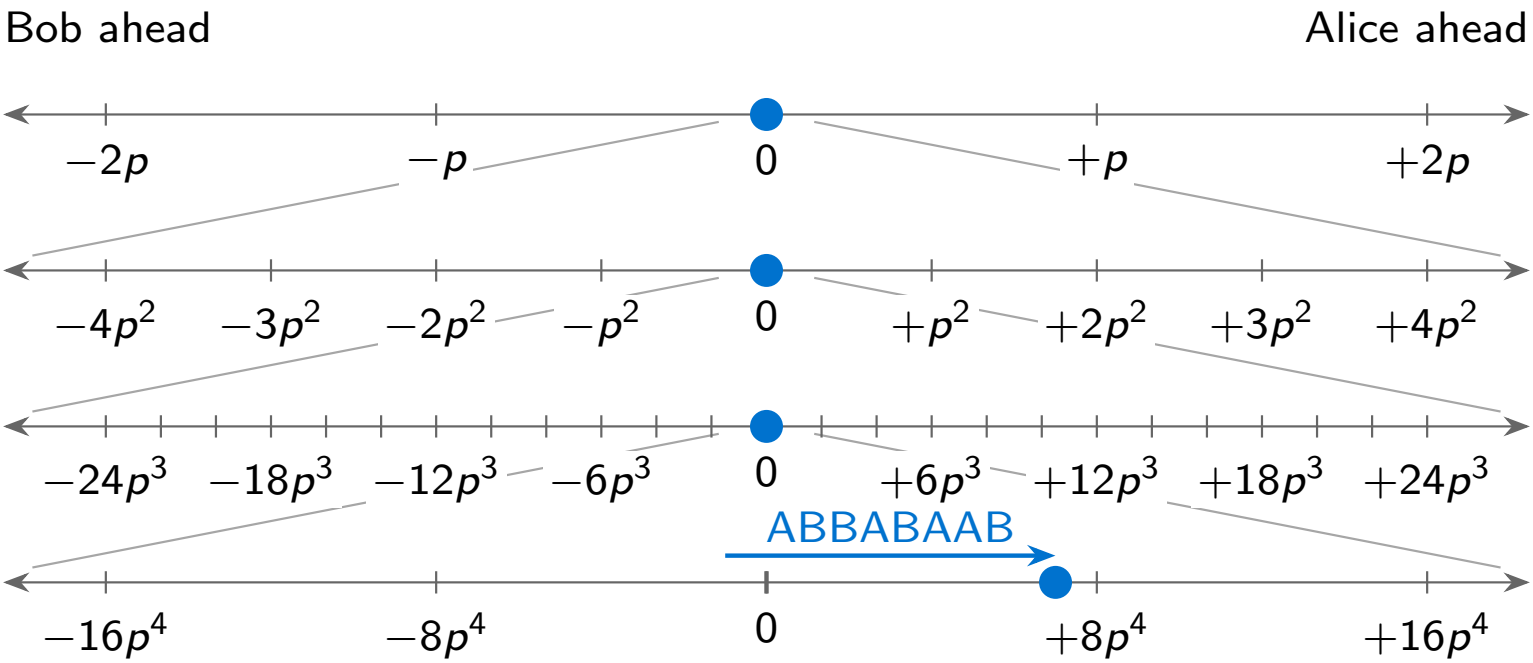
- Negative advantage forces ABBABAAB

  p^4 term: $-p^4 + 8p^4 = 7p^4$

  $\mathrm{AA}\,(\mathrm{B})^2\,(\mathrm{BA})^4\,(\mathrm{BAAB})^{12}$

  $(\mathrm{BAABABBA})^{91}$

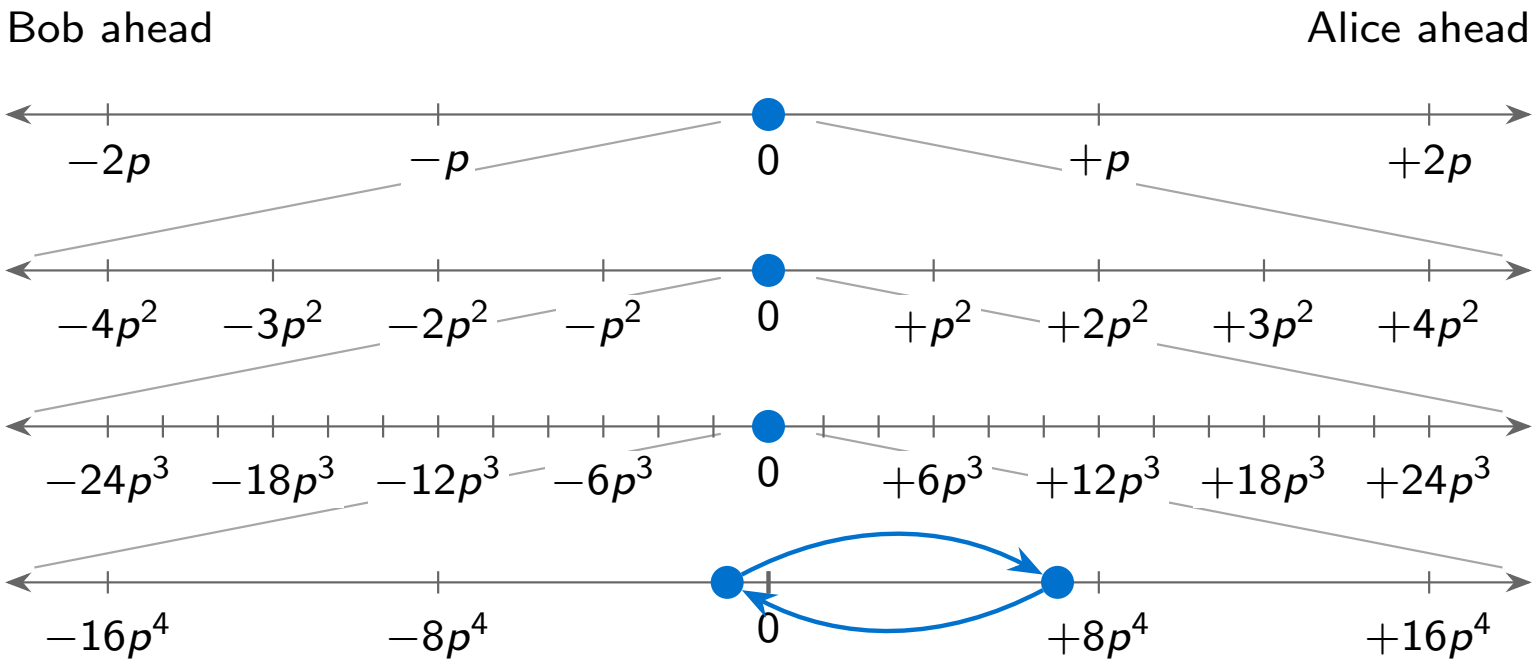
  BAABABBA ABBABAAB



The repeating tail

- The leading term repeats, so the same block choices repeat

$$\mathrm{AA}\,(\mathrm{B})^{2}\,(\mathrm{BA})^{4}\,(\mathrm{BAAB})^{12}\,(\mathrm{BAABABBA})^{91}\,(\mathrm{BAABABBA\ ABBABAAB})^{\infty}$$



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Thank You!

Questions or comments?

Try your own starting sequence



`bit.ly/duel-false-start`

Read the paper: `bit.ly/duel-false-start-paper`

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