Exact Promotion Order of All Two-Row Standard Young Tableaux

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PRIMES-USA

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Background

Definition (Standard Young Tableaux)

A standard Young tableau T is a set S of cells with left-aligned rows and top-aligned columns and a bijection f from S to $\{1, 2, \ldots, |S|\}$ such that

- Rows increase from left to right under f
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For a standard Young tableau T with n cells, the promotion of T is the Young tableau formed in the following way:

- Delete the number in the top left cell
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- Decrease all numbers by 1, and add n into the blank cell

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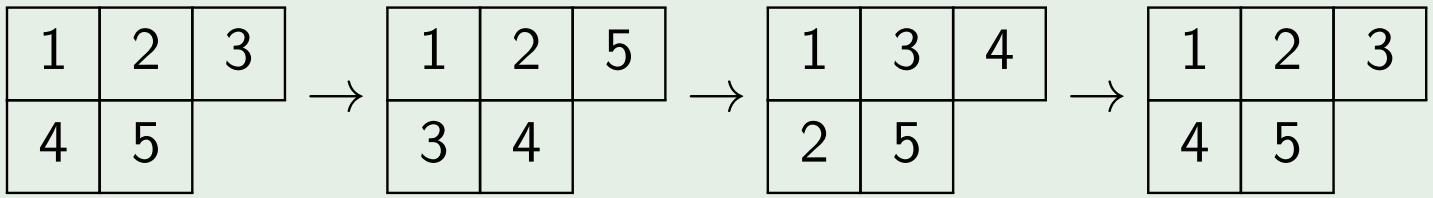
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Therefore, the promotion orders of the above Young tableaux are all equal to 3.

Motivation

Theorem (Haiman, 1992)

For any $m \times n$ rectangular Young tableau, the promotion order divides mn.

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Question

For a two-row standard Young tableau T, what is its promotion order?

Main Findings

Theorem (Z., 2026)

For a given two-row Young tableau T with n cells, we can extract certain d-dimensional vectors $v_1, v_2, \ldots, v_d$ and a positive integer D such that the promotion order of T is exactly

$$\frac{1}{D} \cdot \frac{\begin{vmatrix} - & v_1 & - \\ - & v_2 & - \\ & \vdots & \\ - & v_d & - \end{vmatrix}}{\gcd\left(\begin{vmatrix} 1 & \cdots & 1 \\ - & v_2 & - \\ & \vdots & \\ - & v_d & - \end{vmatrix}, \begin{vmatrix} - & v_1 & - \\ 1 & \cdots & 1 \\ & \vdots & \\ - & v_d & - \end{vmatrix}, \ldots, \begin{vmatrix} - & v_1 & - \\ & \vdots & \\ - & v_{d-1} & - \\ 1 & \cdots & 1 \end{vmatrix} \right)}.$$

Run Length Preservation Under Promotion

Definition (Path)

For a two-row standard Young tableau T with n cells, the path of T is defined through the following process.

- Start from $(0, 0)$, and let k range through $n, n-1, \ldots, 1$ in order
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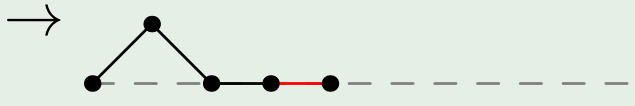
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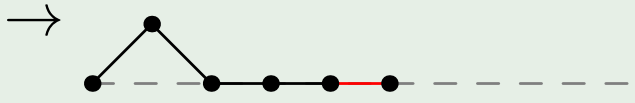
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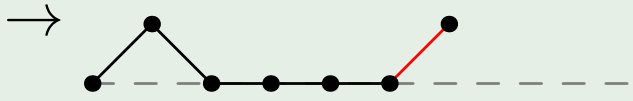
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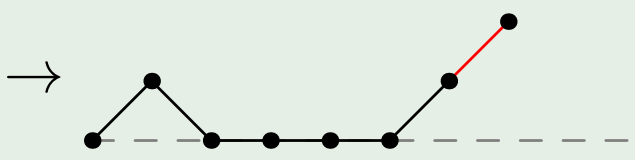
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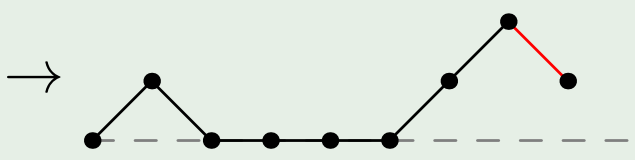
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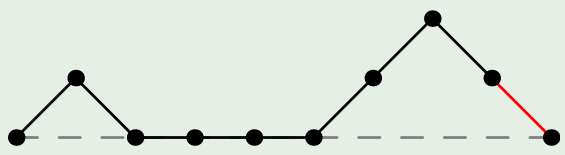
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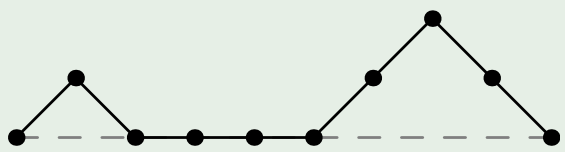
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Run Length Preservation Under Promotion

Definition (Fake Run Lengths)

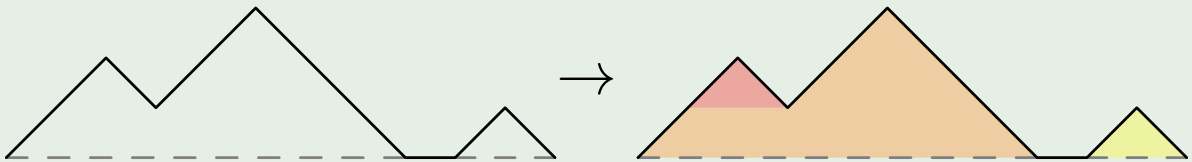
For a two-row standard Young tableau T with a given path, we define the fake run lengths of T by repeatedly removing the smallest peaks consisting of ℓ diagonally right and up units followed by ℓ diagonally right and down units not contained in any other such peak, and adding ℓ as a run length.

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Example



The fake run lengths are $3, 1$, and 1.

Run Length Preservation Under Promotion

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For a two-row standard Young tableau T, we define the run lengths of T as the lexicographically largest fake run lengths of the tableaux in the orbit of T under promotion, when the fake run lengths are arranged in decreasing order.

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Lemma (Z., 2026)

The set of tableaux where the fake run lengths are the same as the run lengths is preserved under a promotion step operation which involves sufficiently many promotions.

Promotion Order

Definition (Peak Positions)

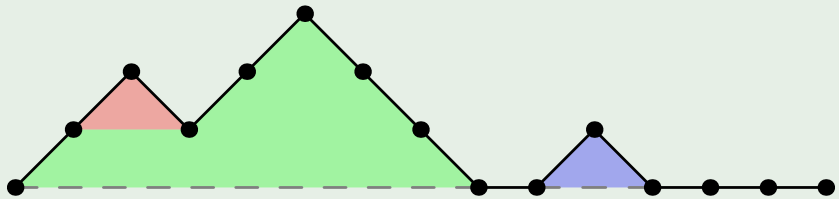
On the path of a tableau, for any peak of 2ℓ diagonal units by deleting the $2\min(\ell, x)$ center diagonal units from every other peak, where there are $2x$ diagonal units in the other peak. After this, we define the position of the peak as the number of units to the right of the right endpoint of the peak.

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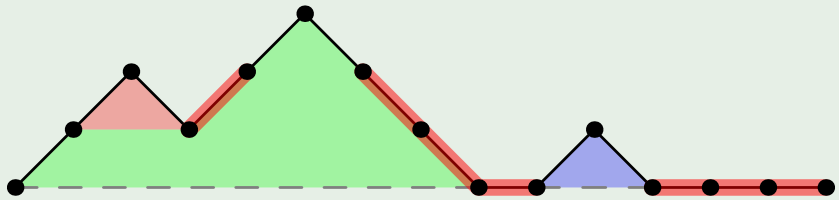
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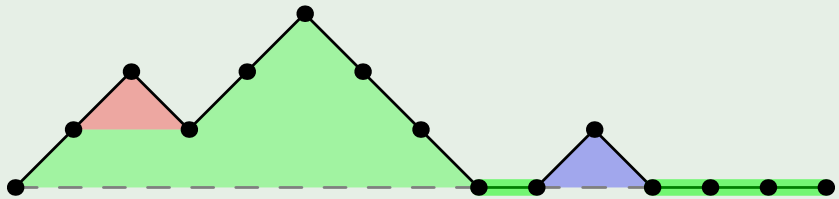
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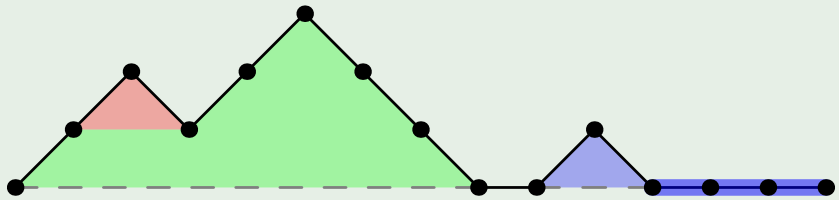
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Example



The red, green, and blue peak positions are $7, 4$, and 3.

Promotion Order

Definition (Peak Position Process)

For d-dimensional vectors $v_1, v_2, \ldots, v_d$ of positive integers and v a d-dimensional vector of nonnegative reals, we define a process consisting of steps of the following form.

- Decrease all coordinates of v by 1.
- Add v_k to v for all k with minimal and negative kth coordinate.

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Lemma (Z., 2026)

For T a two-row standard Young tableau, we can choose $v_1, v_2, \ldots, v_d$ such that the tuples of peak positions match the process defined above.

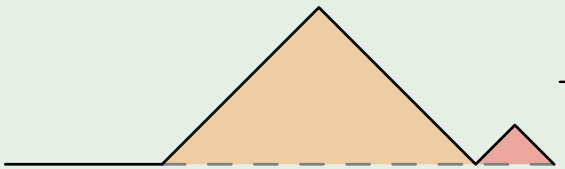
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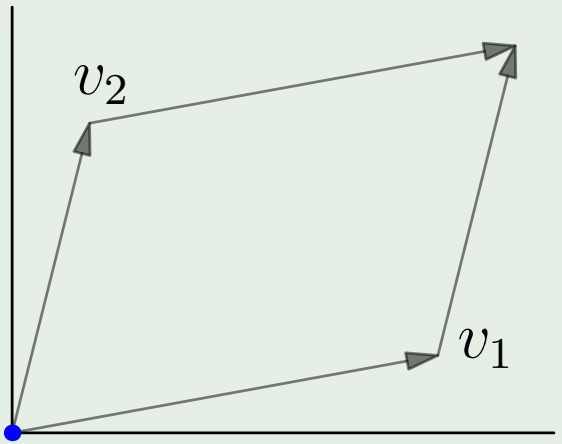
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2	7	8	9	10				

$\rightarrow$  $\rightarrow (0, 0)$

$\rightarrow$ 

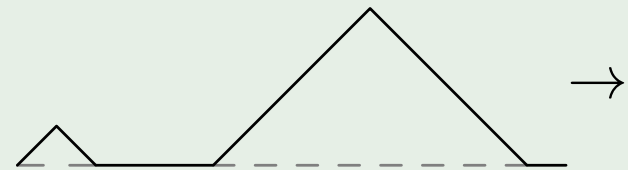
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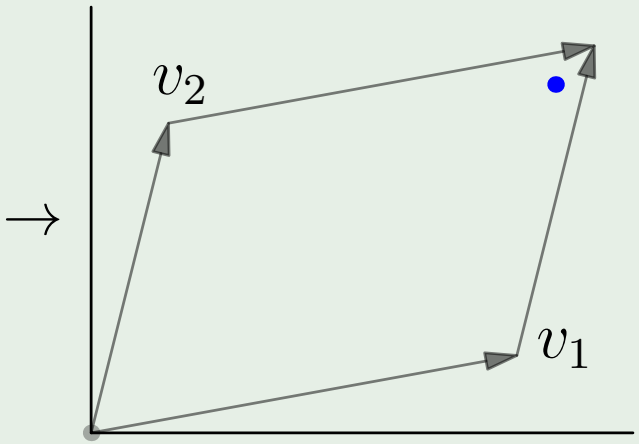
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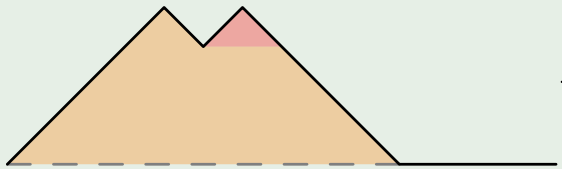
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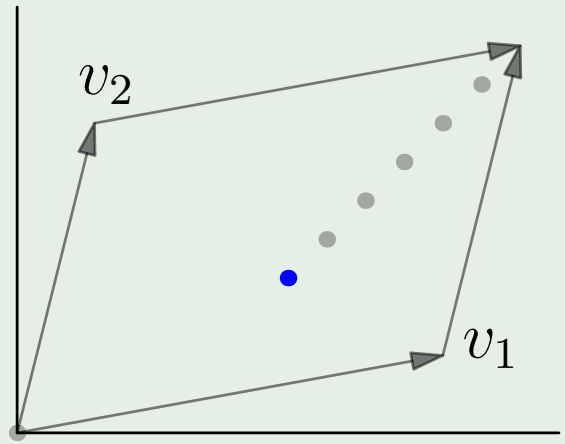
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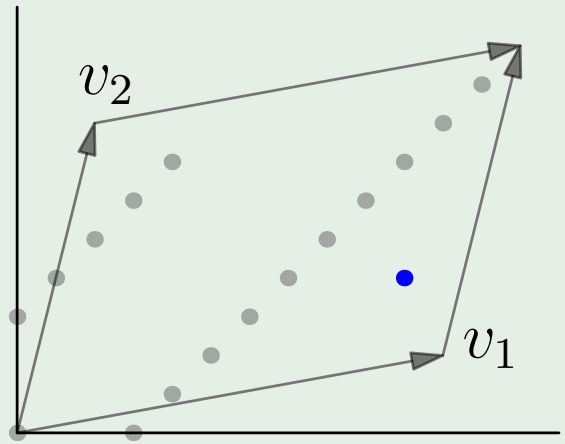
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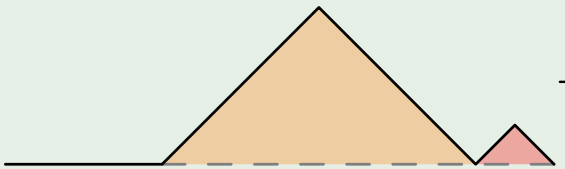
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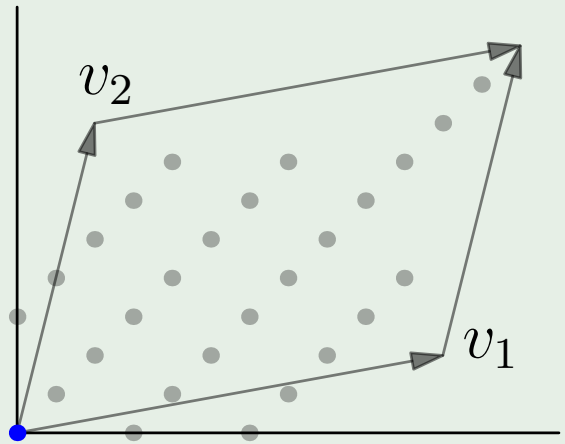
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For T a two-row standard Young tableau, we can choose $v_1, v_2, \ldots, v_d$ such that the tuples of peak positions match the process defined above.

Example

1	3	4	5	6	11	12	13	14
2	7	8	9	10				

$\rightarrow$  $\rightarrow (0, 0)$

$\rightarrow$ 

Promotion Order

Lemma (Z., 2026)

For this process, the period P is exactly

$$\frac{\begin{vmatrix} - & v_1 & - \\ - & v_2 & - \\ & \vdots & \\ - & v_d & - \end{vmatrix}}{\gcd\left(\begin{vmatrix} 1 & \cdots & 1 \\ - & v_2 & - \\ & \vdots & \\ - & v_d & - \end{vmatrix}, \begin{vmatrix} - & v_1 & - \\ 1 & \cdots & 1 \\ & \vdots & \\ - & v_d & - \end{vmatrix}, \ldots, \begin{vmatrix} - & v_1 & - \\ & \vdots & \\ - & v_{d-1} & - \\ 1 & \cdots & 1 \end{vmatrix}\right)}.$$

Promotion Order

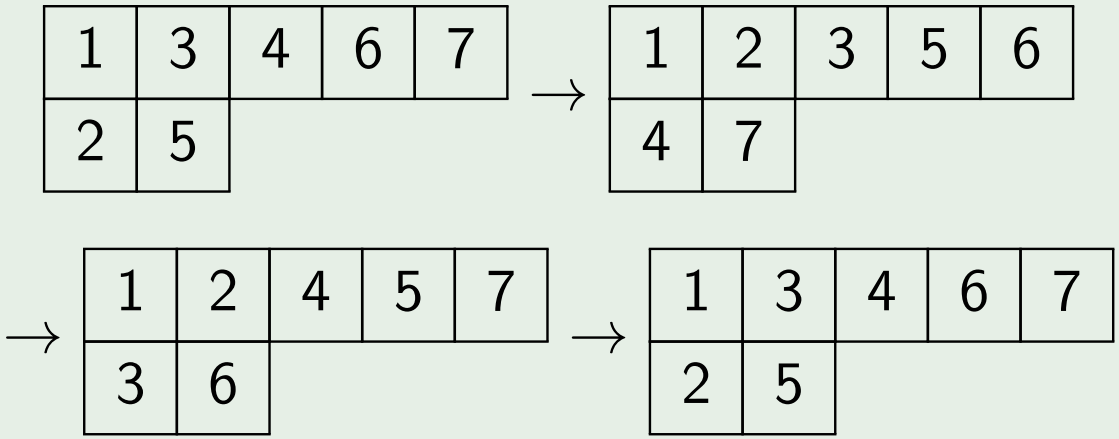
Theorem (Z., 2026)

For a given two-row Young tableau T with n cells, we can extract certain d-dimensional vectors $v_1, v_2, \ldots, v_d$ and a positive integer D such that the promotion order of T is exactly

$$\frac{1}{D} \cdot \frac{\begin{vmatrix} - & v_1 & - \\ - & v_2 & - \\ & \vdots & \\ - & v_d & - \end{vmatrix}}{\gcd\left(\begin{vmatrix} 1 & \cdots & 1 \\ - & v_2 & - \\ & \vdots & \\ - & v_d & - \end{vmatrix}, \begin{vmatrix} - & v_1 & - \\ 1 & \cdots & 1 \\ & \vdots & \\ - & v_d & - \end{vmatrix}, \ldots, \begin{vmatrix} - & v_1 & - \\ & \vdots & \\ - & v_{d-1} & - \\ 1 & \cdots & 1 \end{vmatrix} \right)}.$$

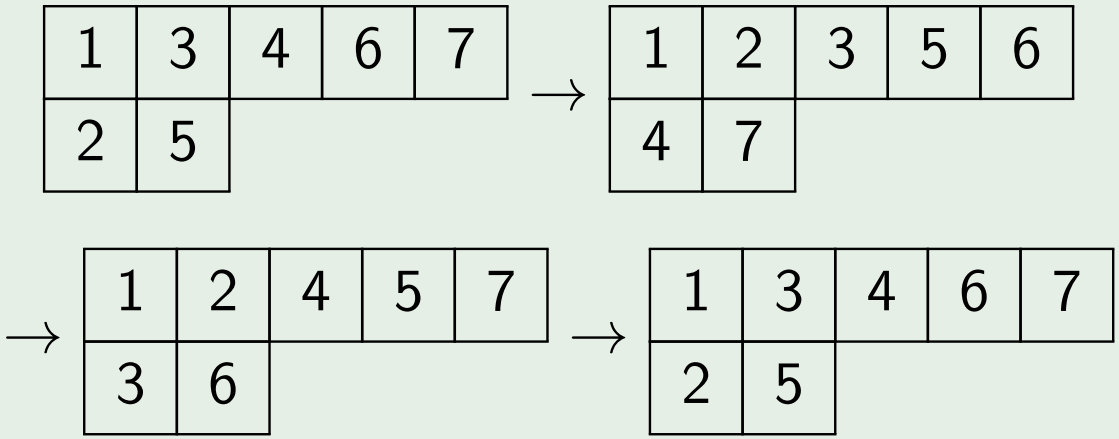
Promotion Order

Example



Promotion Order

Example



We have that $D = 2$ here. The tuples of positions of peaks cycle

$$(0,1) \to (3,2) \to (2,1) \to (1,0) \to (2,3) \to (1,2) \to \cdots$$

and so the period of these tuples is 6. The promotion order is then $\frac{1}{2} \cdot 6 = 3$.

Promotion Order

- We can find the following result of Striker (2017) using this theorem.

Example

The least common multiple of the promotion orders over all two-row Young tableaux with 8 cells in the first row and 6 in the second is 7554844752.

Promotion Order

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Example

The least common multiple of the promotion orders over all two-row Young tableaux with 8 cells in the first row and 6 in the second is 7554844752.

Proof.

Run Lengths	Promotion Order
(6)	8
$(5,1)$	73
$(4,2)$	16
$(4,1,1)$	58

Promotion Order

Proof.

Run Lengths	Promotion Order
$(3,2,1)$	114
$(3,1,1,1)$	43
$(2,2,2)$	12
$(2,2,1,1)$	56
$(2,1,1,1,1)$	28
$(1,1,1,1,1,1)$	13

Promotion Order

Proof.

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Therefore, the least common multiple of the promotion orders is

$$\begin{aligned}\operatorname{lcm}(8,73,16,58,114,43,12,56,28,13) &= 2^4 \cdot 3 \cdot 7 \cdot 13 \cdot 19 \cdot 29 \cdot 43 \cdot 73 \\ &= 7554844752.\end{aligned}$$

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Thank You!

The End