The Forest Monk Poset

How multiplication creates a combinatorial order

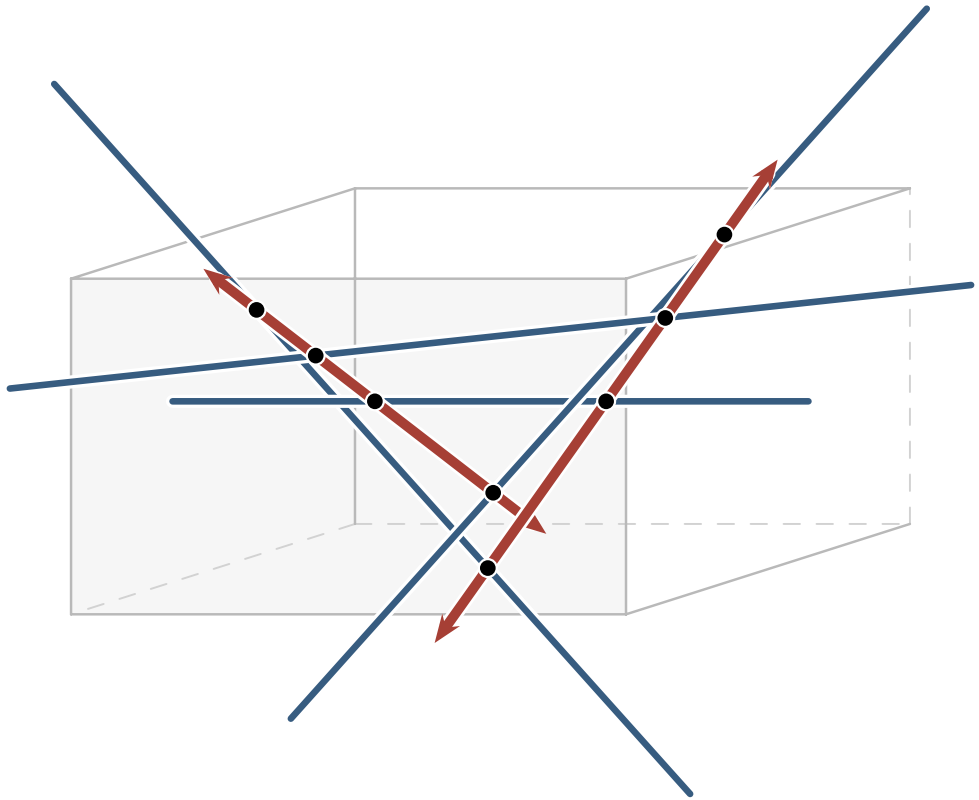
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MIT PRIMES Conference
October 24, 2026

A geometric counting problem

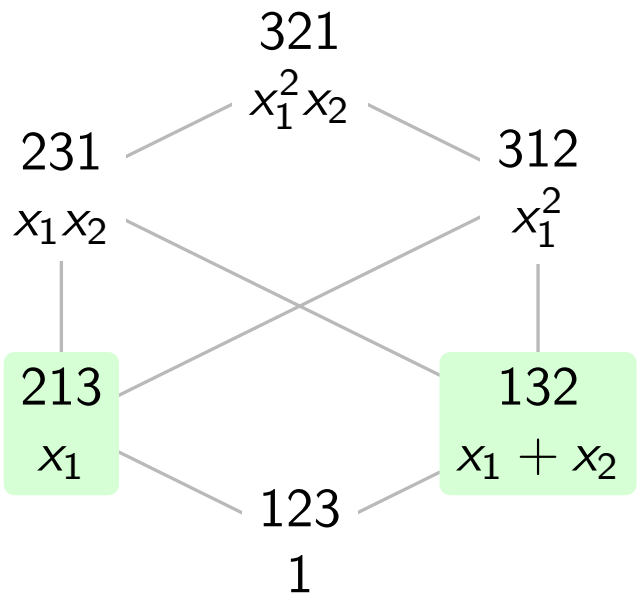
In the nineteenth century, Hermann Schubert studied questions such as: how many lines meet four given lines in general position in space?



Each red line meets all four blue lines. Can we count solutions without finding their coordinates?

Schubert polynomials

Schubert polynomials encode geometric conditions. Multiplication records how those conditions intersect.



$$\mathfrak{S}_{213}\mathfrak{S}_{132} = 1 \cdot \mathfrak{S}_{312} + 1 \cdot \mathfrak{S}_{231}.$$

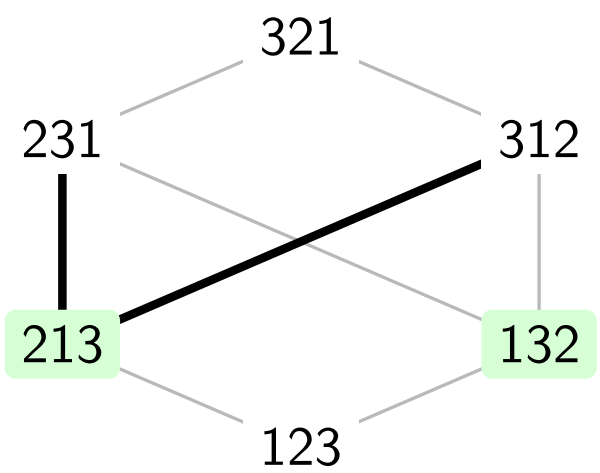
The coefficients are *structure constants*. A general positive combinatorial rule for them is still unknown.

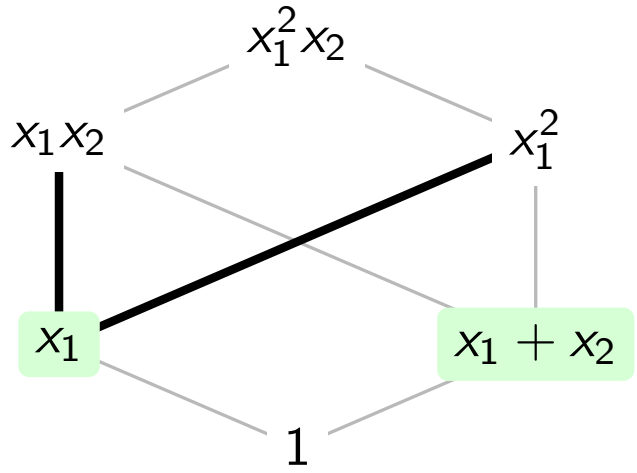
Monk's rule reveals Bruhat order

Theorem (Monk, 1959)

The expansion of $(x_1 + \cdots + x_i)\mathfrak{S}_w$ *is determined by a combinatorial rule on permutations.*
Its summands are distinct: every nonzero coefficient is 1.

$$\mathfrak{S}_{213}(x_1 + x_2) = 1 \cdot \mathfrak{S}_{312} + 1 \cdot \mathfrak{S}_{231}.$$





Each output gives one upward step. Together, these steps give *Bruhat order*.

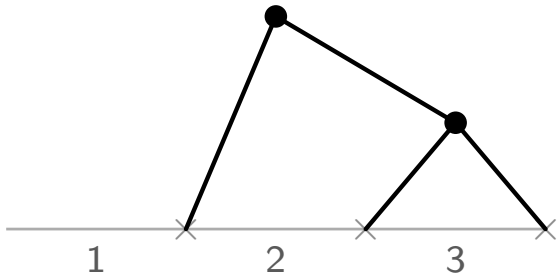
Why forest polynomials?

Nadeau and Tewari (2023) introduced *forest polynomials*. Binary forests index them, just as permutations index Schubert polynomials.

- Schubert polynomials expand as nonnegative sums of forest polynomials.
- Geometry: introduced to study the *permutahedral variety*.
- Connections to *quasisymmetric functions*.
- Parking functions: related parking procedures give counting rules.

Indexed forests

Here is one of the binary forests that indexes a polynomial.



Numbers label the gaps between leaves, not the leaves.

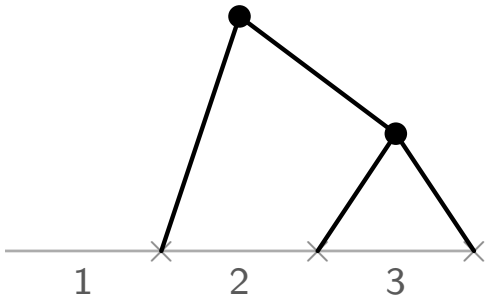
Definition (Support)

The *support* of a forest is the set of indices spanned by its trees. Here, $\text{supp}(F) = \{2, 3\}$.

This forest has 2 internal vertices.

From forests to polynomials

A combinatorial recipe assigns a polynomial P_F to each forest F.



$$\longrightarrow \quad P_F = x_1x_2 + x_1x_3 + x_2x_3$$

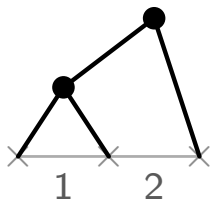
The shape and gap indices determine the polynomial.

What we will use

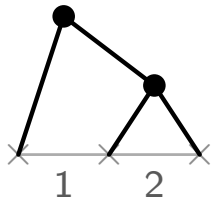
Forest polynomials form a basis: every product can be written uniquely as a linear combination of them.

A small forest window

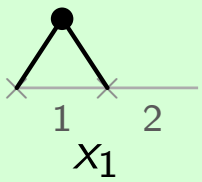
Restrict to gaps 1 and 2: this finite *window* has five forests.



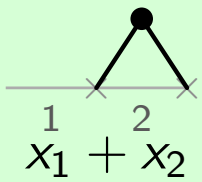
x_1^2



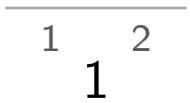
$x_1 x_2$



x_1



$x_1 + x_2$



1

The empty forest gives 1.

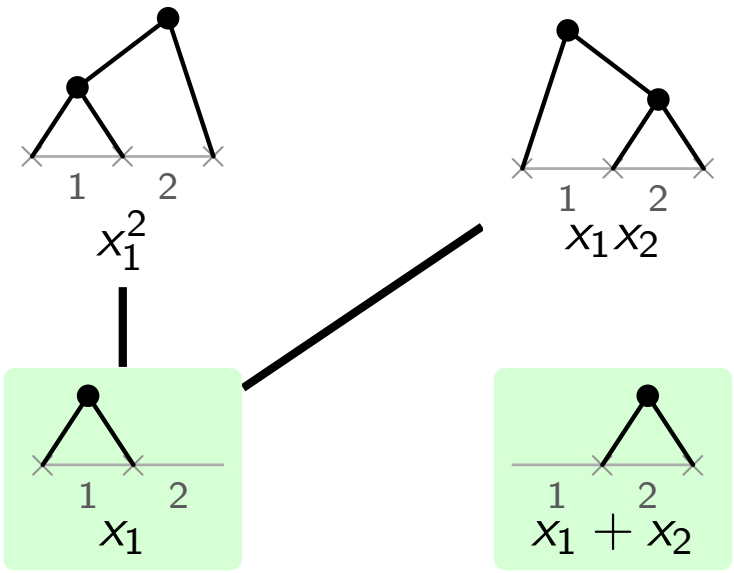
One internal vertex gives a linear polynomial.

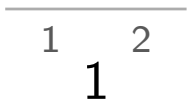
Two internal vertices give a quadratic polynomial.

The green forests give the same linear factors as before.

Multiplication gives upward steps

Start with the forest whose polynomial is x_1.





Multiply by the other green polynomial:

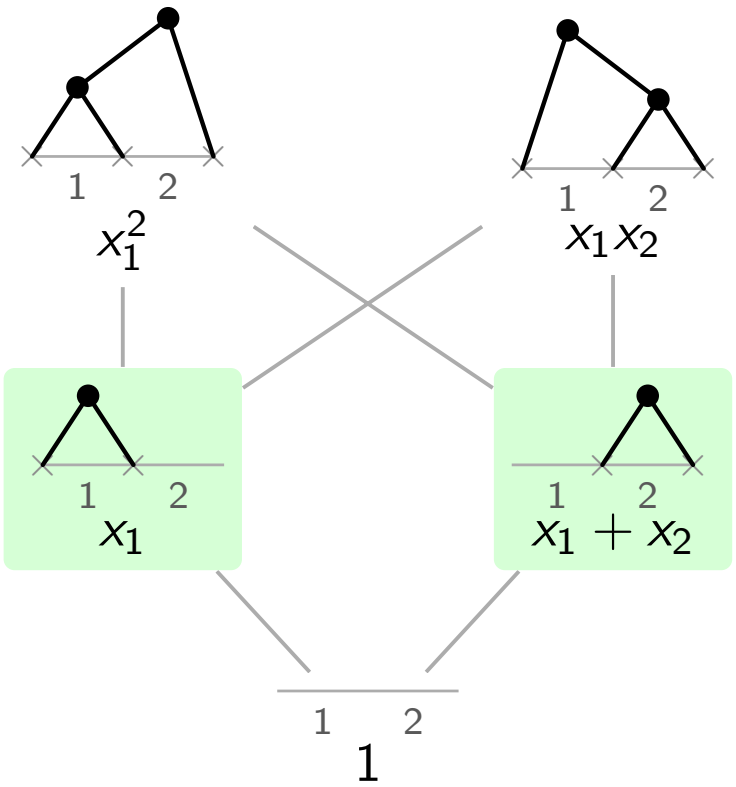
$$x_1(x_1 + x_2)$$
$$= x_1^2 + x_1x_2.$$

Each summand gives one upward step.

The two structure constants here are both 1.

The forest Monk poset

Use every linear factor $x_1 + \cdots + x_i$ to create upward steps.



Forest Monk rule

$$(x_1 + \cdots + x_i)P_F = \sum_G P_G,$$

with distinct summands.

Following upward paths defines the *forest Monk poset*.

My work studies the structure of this order.

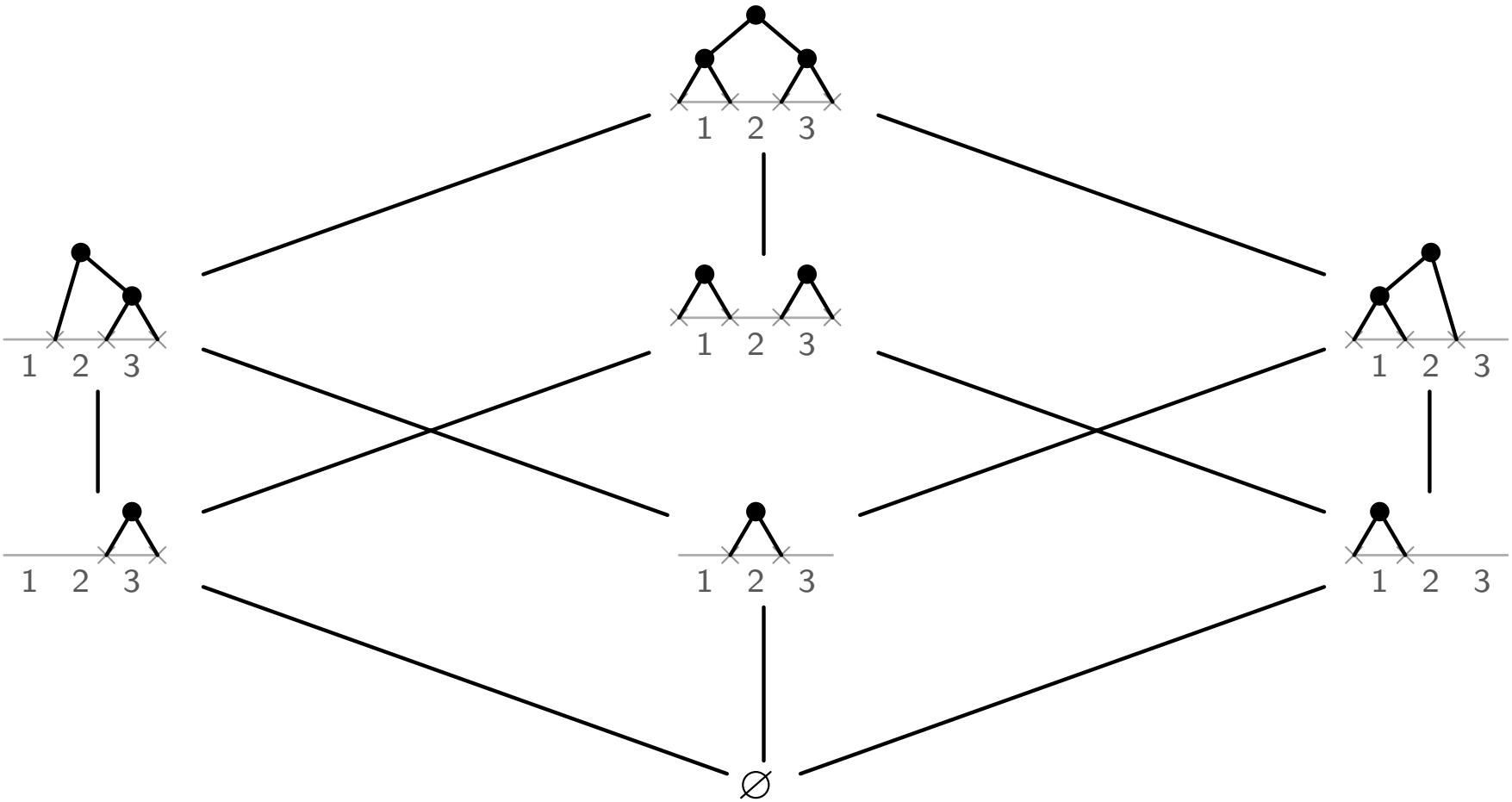
Each upward step increases the number of internal vertices by one.

Questions I researched

1. Which familiar shapes and lattices appear?
   Theorem 1: Boolean intervals.
2. How large can families of incomparable forests be?
   Theorem 2: Strong Sperner, proved using chains.
3. In a fixed window, how many forests have k internal vertices?
   Counting formula in the appendix.
4. What can geometry tell us about this order?
   Open question: the shellability conjecture.

A lower interval

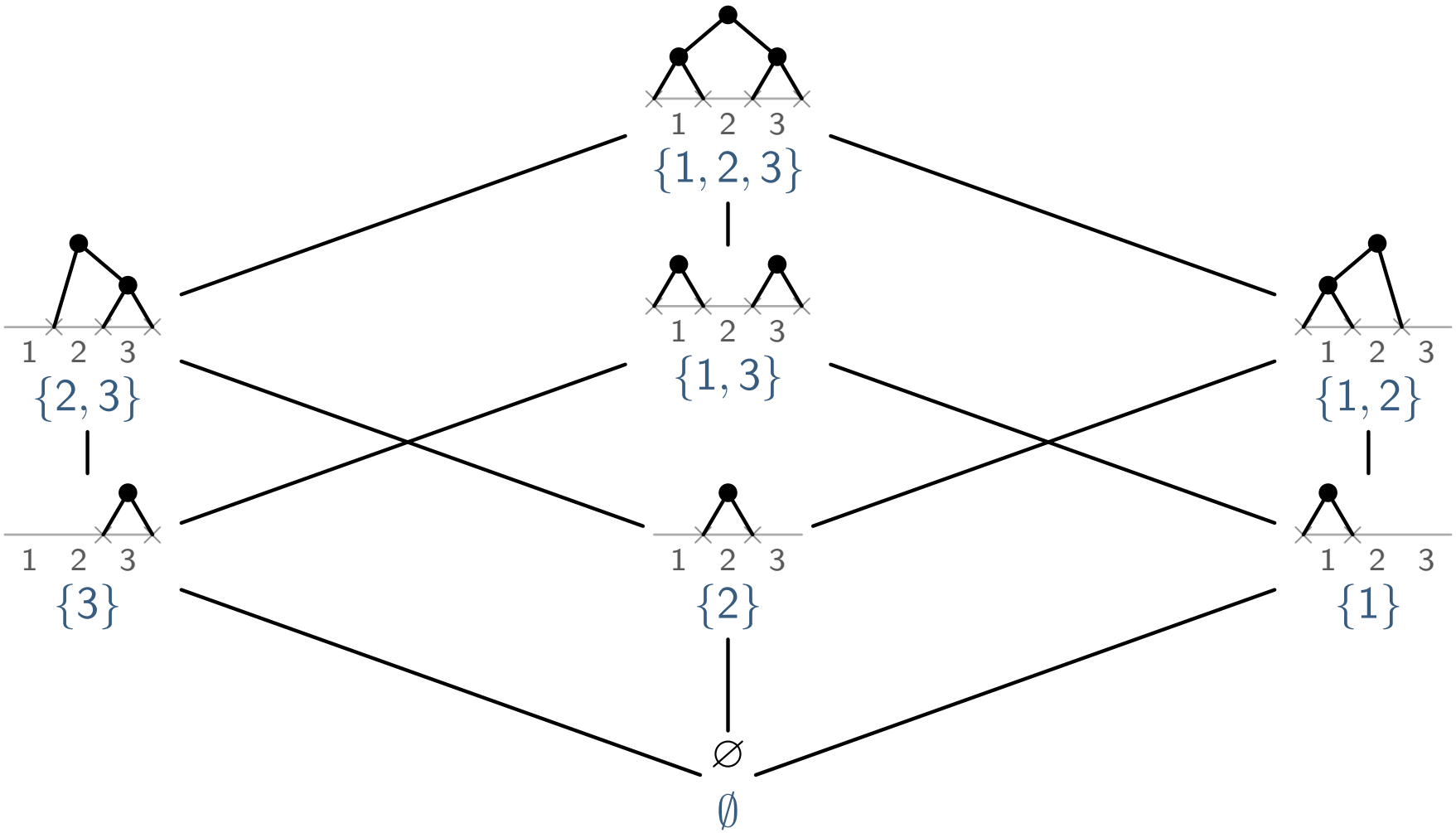
On three gaps, fix a top forest F. Keep every forest with an upward path to F.



These eight forests form the *lower interval* $[\varnothing, F]$.

The cube is a Boolean lattice

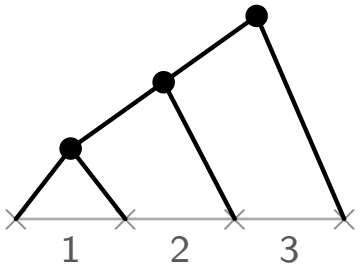
The blue sets record the *support* of each forest.



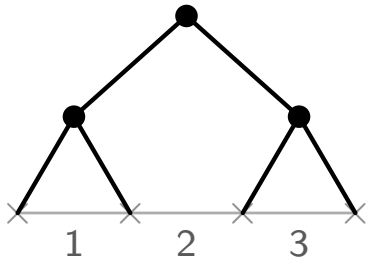
Subsets of $\{1, 2, 3\}$, ordered by inclusion: a *Boolean lattice*.

Theorem 1: Boolean intervals

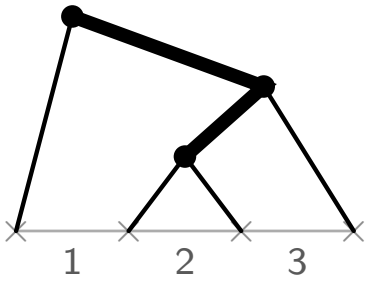
My first result identifies exactly which top forests give this cube behavior.



one straight arm



two straight arms



a **bent corner**

In an *up-down tree*, the internal vertices form a straight left arm and a straight right arm from the root; either arm may be empty.

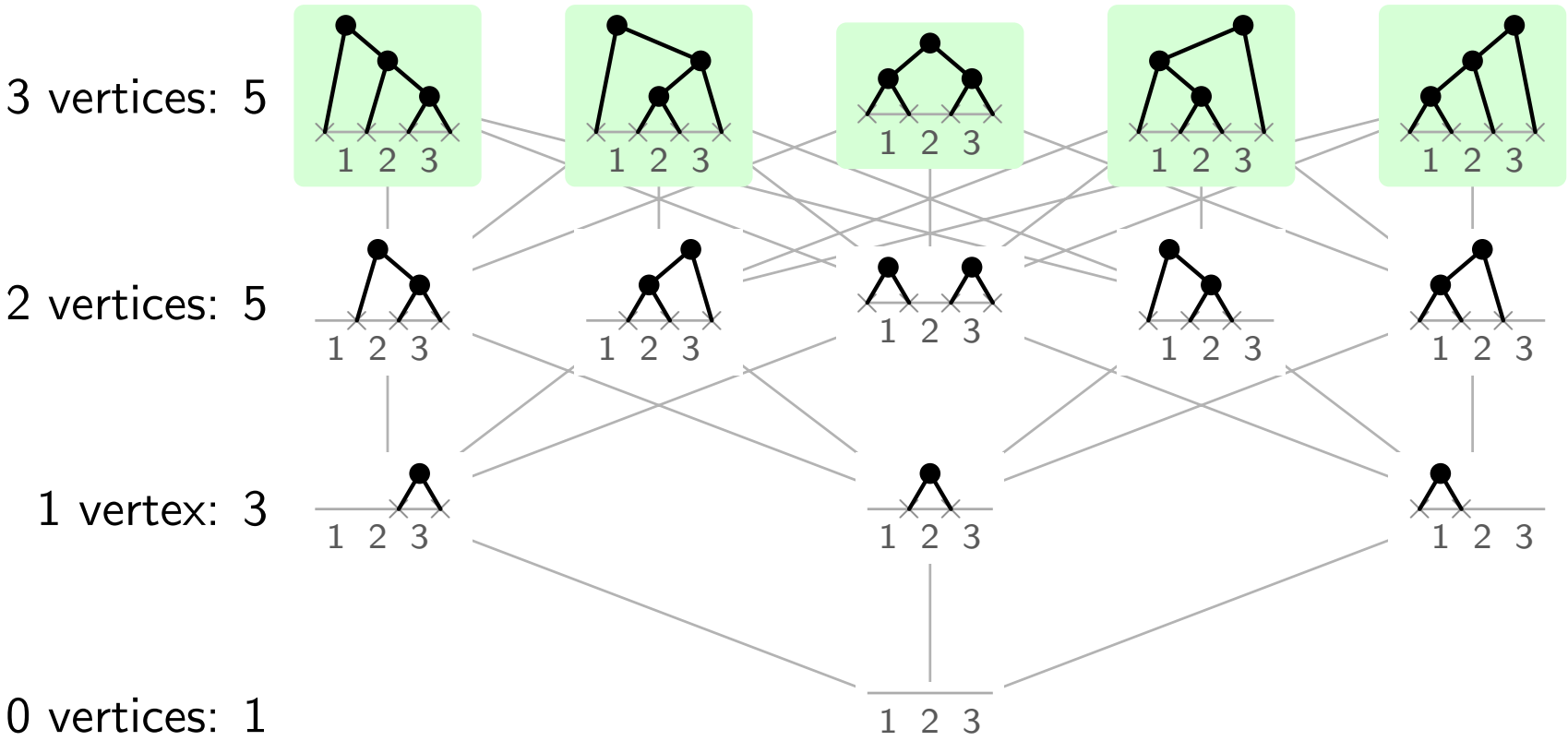
Theorem (Boolean classification)

$[\varnothing, F]$ *is a Boolean lattice if and only if* F *has no* **bent corner**.

A forest analogue of Tenner's pattern-avoidance criterion.

How large can an antichain be?

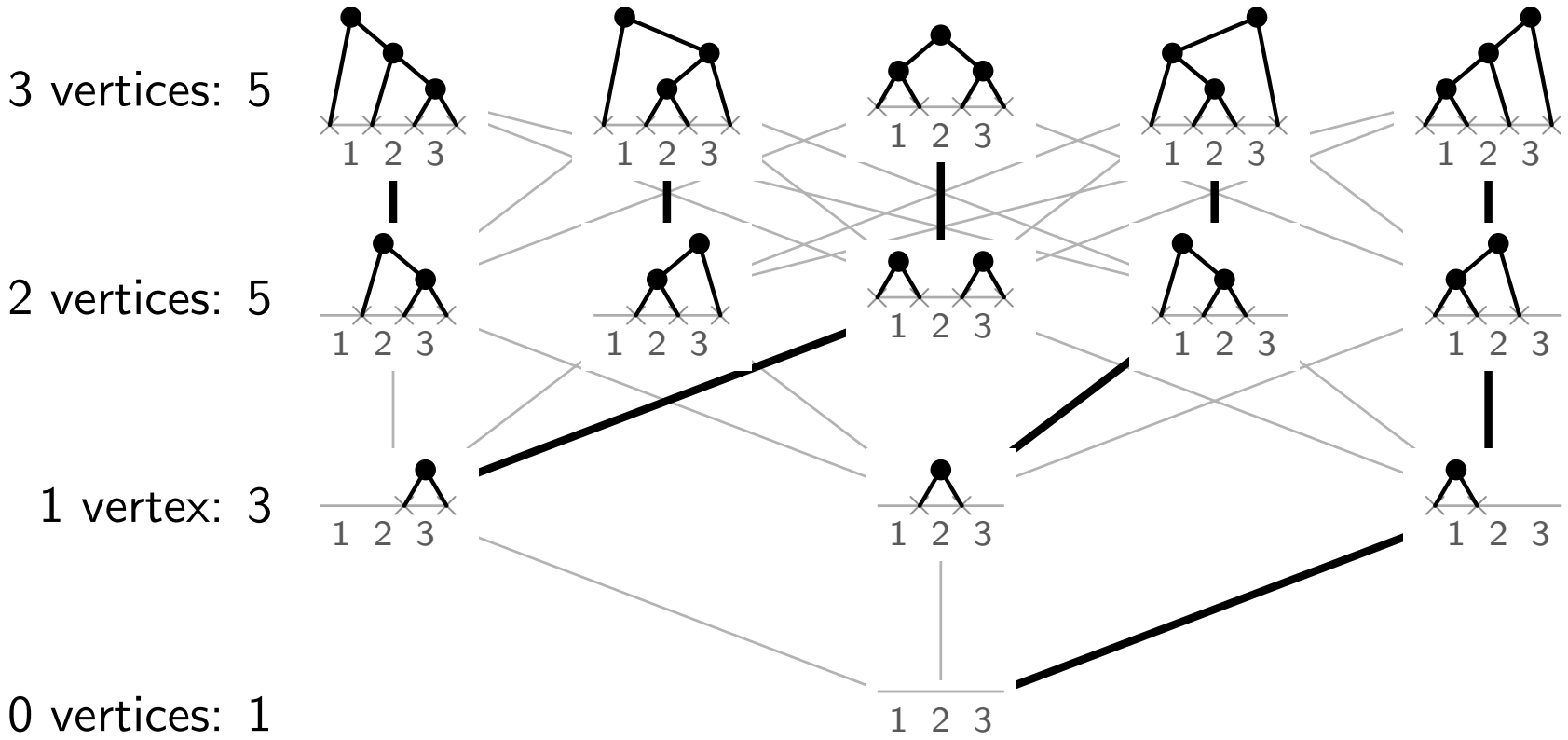
All 14 forests on three gaps, grouped by number of internal vertices.



The five green forests form an *antichain*: none lies above another.
Can mixing rows give an antichain with more than 5 forests?

Chains bound antichains

Every forest belongs to exactly one bold path, or *chain*.



An antichain uses at most one forest per chain: at most 5.
Two antichains use at most 10, attained by the two top rows.

Theorem 2: Strong Sperner

Each row groups forests with the same number of internal vertices.

Definition (Strong Sperner)

A finite forest window is *strongly Sperner* if, for each r up to its number of rows, a union of r antichains is no larger than its r largest rows combined.

Theorem (Strong Sperner for forests)

For every n, the forests supported in gaps $1, \ldots, n$ form a strongly Sperner poset.

The same chain argument works in every finite window.

The largest rows attain the bound.

Two structural results

Multiplication creates an order with recognizable structure.

Boolean intervals

Avoiding a **bent corner** makes the entire lower interval a Boolean lattice.

Strong Sperner

In each finite window, the largest rows give the largest unions of antichains.
Disjoint chains explain the bound.

These extend two familiar features of Bruhat order to forests.

Next question: shellability

Chains turn an order into a geometric object. *Shellability* asks whether its pieces can be assembled in a controlled way that helps us understand its shape.

Conjecture (Shellability)

Every principal lower interval $[\varnothing, F]$ *in the forest Monk poset is shellable.*

Forest polynomials have geometry via the *quasisymmetric flag variety*.
What can that geometry explain about this new order?

The hope: insights from forests and Schubert calculus will inform each other.

Acknowledgments

I would like to thank:

- Dora Woodruff, my mentor, for her guidance and support.
- The MIT PRIMES program for making this research possible.

Thank you for listening.

Sources

[1] D. Monk. *The geometry of flag manifolds*. Proc. London Math. Soc., 1959.

[2] P. Nadeau and V. Tewari. *Forest polynomials and the class of the permutahedral variety*. arXiv:2306.10939, 2023; Adv. Math., 2024.

[3] P. Nadeau, H. Spink, and V. Tewari. *Quasisymmetric divided differences*. arXiv:2406.01510v3, 2026, Theorem 7.4.

[4] B. E. Tenner. *Pattern avoidance and the Bruhat order*. J. Combin. Theory Ser. A, 2007.

[5] R. P. Stanley. *Weyl groups, the hard Lefschetz theorem, and the Sperner property*. SIAM J. Algebraic Discrete Methods, 1980.

[6] N. Bergeron, L. Gagnon, P. Nadeau, H. Spink, and V. Tewari. *The quasisymmetric flag variety: a toric complex on noncrossing partitions*. Forum Math. Pi, 2026; arXiv:2508.12171.

Appendix: counting forests

Restrict to forests supported in gaps $1, \ldots, n$.

Theorem (Counting by internal vertices)

The number of forests with exactly k internal vertices is

$$\frac{n+1-k}{n+1}\binom{n+k}{k}, \qquad 0 \le k \le n.$$

For $n = 3$, the counts for $k = 0, 1, 2, 3$ are $1, 3, 5, 5$.

For $n = 6$, the counts for $k = 0, \ldots, 6$ are $1, 6, 20, 48, 90, 132, 132$.

These are ballot numbers. The chain partition explains why the largest rows give the largest unions of antichains.