A tour of the generalized Catalan numbers

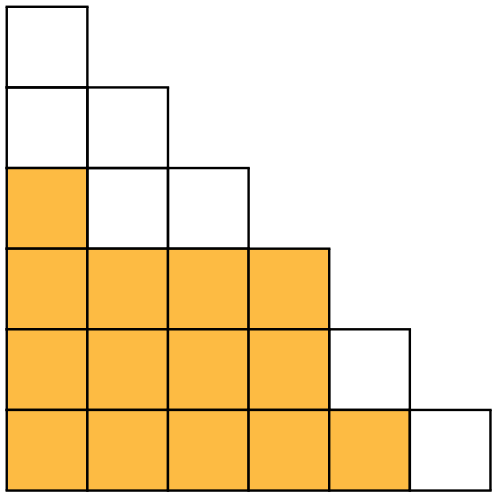
Michael Luo
mentor: Xinchun Ma

Minnetonka High School

MIT PRIMES conference
October 24, 2026

The Catalan numbers

$$C_n = \frac{1}{n+1}\binom{2n}{n} = \sum_{\lambda \subseteq \rho} 1$$



$$C_7 = 429$$

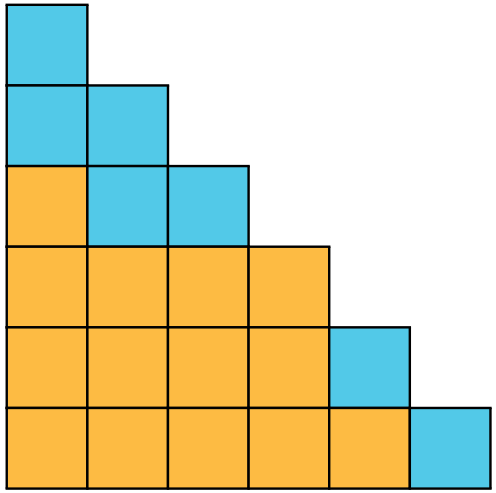
A very large number of combinatorial interpretations are known [...] This is probably the longest entry in the OEIS, and rightly so.

—OEIS A000108

The q-Catalan numbers

$$\text{area}(\lambda) = \binom{n}{2} - |\lambda| \qquad C_n(q) = \sum_{\lambda \subseteq \rho} q^{\text{area}(\lambda)}$$

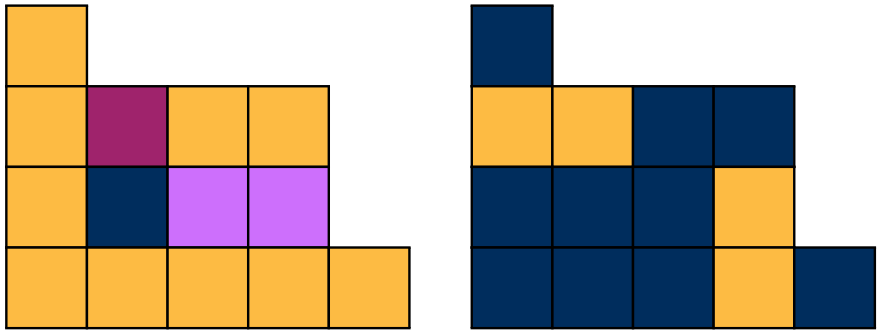
Contributes q^7 :



$$\begin{aligned} C_7(q) = {} & q^{21} + q^{20} + 2q^{19} + 3q^{18} + 5q^{17} + 7q^{16} + 11q^{15} + 13q^{14} + 18q^{13} \\ & + 22q^{12} + 28q^{11} + 32q^{10} + 37q^9 + 40q^8 + 44q^7 + 43q^6 + 40q^5 \\ & + 35q^4 + 25q^3 + 15q^2 + 6q + 1 \end{aligned}$$

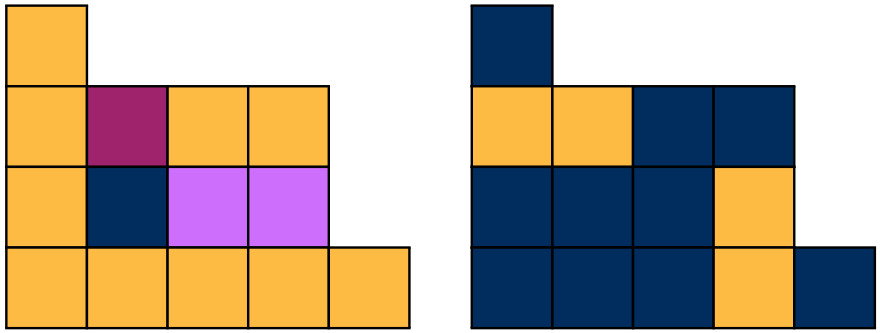
The q, t-Catalan numbers

$$\mathrm{dinv}(\lambda) = \#\{c \in \lambda : \mathrm{arm}(c) - \mathrm{leg}(c) = 0, 1\}$$



The q, t-Catalan numbers

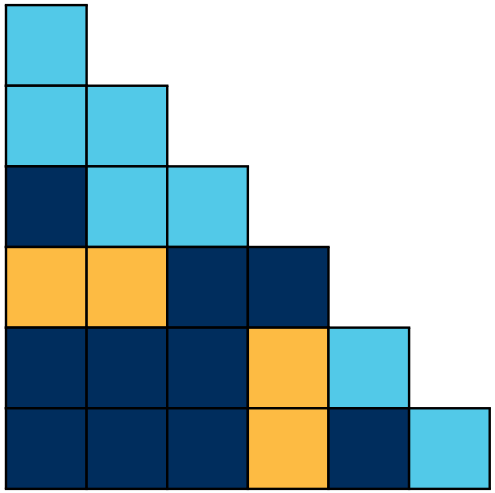
$$\mathrm{dinv}(\lambda) = \#\{c \in \lambda : \mathrm{arm}(c) - \mathrm{leg}(c) = 0, 1\}$$



$$C_n(q,t) = \sum_{\lambda \subseteq \rho} q^{\mathrm{area}(\lambda)} t^{\mathrm{dinv}(\lambda)}$$

(Garsia–Haglund 2002)

Contributes $q^7 t^{10}$:



The q, t-Catalan numbers

$$C_n(q,t) = \sum_{\lambda \subseteq \rho} q^{\text{area}(\lambda)} t^{\text{dinv}(\lambda)}$$

$$C_1(q,t) = 1$$

$$C_2(q,t) = q + t$$

$$C_3(q,t) = q^3 + q^2 t + q t^2 + t^3 + qt$$

$$C_4(q,t) = q^6 + q^5 t + q^4 t^2 + q^3 t^3 + q^2 t^4 + q t^5 + t^6 + q^4 t + q^3 t^2 + q^2 t^3 + q t^4 + q^3 t + q^2 t^2 + q t^3$$

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- $C_n(q,1) = C_n(q)$

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- $C_n(q,1) = C_n(q)$
- Symmetric in q and t
- Polynomial with positive integer coefficients

The generalized q, t-Catalan numbers

$$T = \begin{array}{|c|c|c|}\hline 6 & 7 & \multicolumn{1}{c}{} \\ \hline 3 & 4 & \multicolumn{1}{c}{} \\ \hline 1 & 2 & 5 \\ \hline \end{array}$$

The generalized q, t-Catalan numbers

$$T = \begin{array}{cccc} 2 & 6 & 7 & \\ 1 & 3 & 4 & \\ 0 & 1 & 2 & 5 \\ & 0 & 1 & 2 \end{array}$$

$$(z_1, z_2, z_3, z_4, z_5, z_6, z_7) = (1, q, t, qt, q^2, t^2, q^1 t^2)$$

The generalized q, t-Catalan numbers

$$T = \begin{array}{cccc} \mathbf{2} & 6 & 7 & \\ 1 & 3 & 4 & \\ 0 & 1 & 2 & 5 \\ & 0 & \mathbf{1} & 2 \end{array}$$

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$$\mathrm{wt}(T) = \prod_{i=2}^{n} \frac{1}{(1 - z_i^{-1})(1 - qtz_{i-1}/z_i)} \prod_{i<j} \frac{(1 - z_i/z_j)(1 - qtz_i/z_j)}{(1 - qz_i/z_j)(1 - tz_i/z_j)}$$

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Gorsky–Hawkes–Schilling–Rainbolt 2020:

$$f(a_1, \ldots, a_n) = \sum_{n\text{-cell } T} z_1^{a_1} \ldots z_n^{a_n} \mathrm{wt}(T)$$

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$$f(a_1, \ldots, a_n) = \sum_{n\text{-cell } T} z_1^{a_1} \ldots z_n^{a_n} \mathrm{wt}(T)$$

$$F(a_2, \ldots, a_n) = f(\bullet, a_2, \ldots, a_n)$$

Properties

$$F(a_2, \ldots, a_n) = \sum_T z_2^{a_2} \ldots z_n^{a_n} \operatorname{wt}(T)$$

- $F(\underbrace{1, \ldots, 1}_{n-1}) = f(\underbrace{1, \ldots, 1}_{n}) = C_n(q, t)$

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- Polynomial with integer coefficients
- Coefficients not necessarily positive:

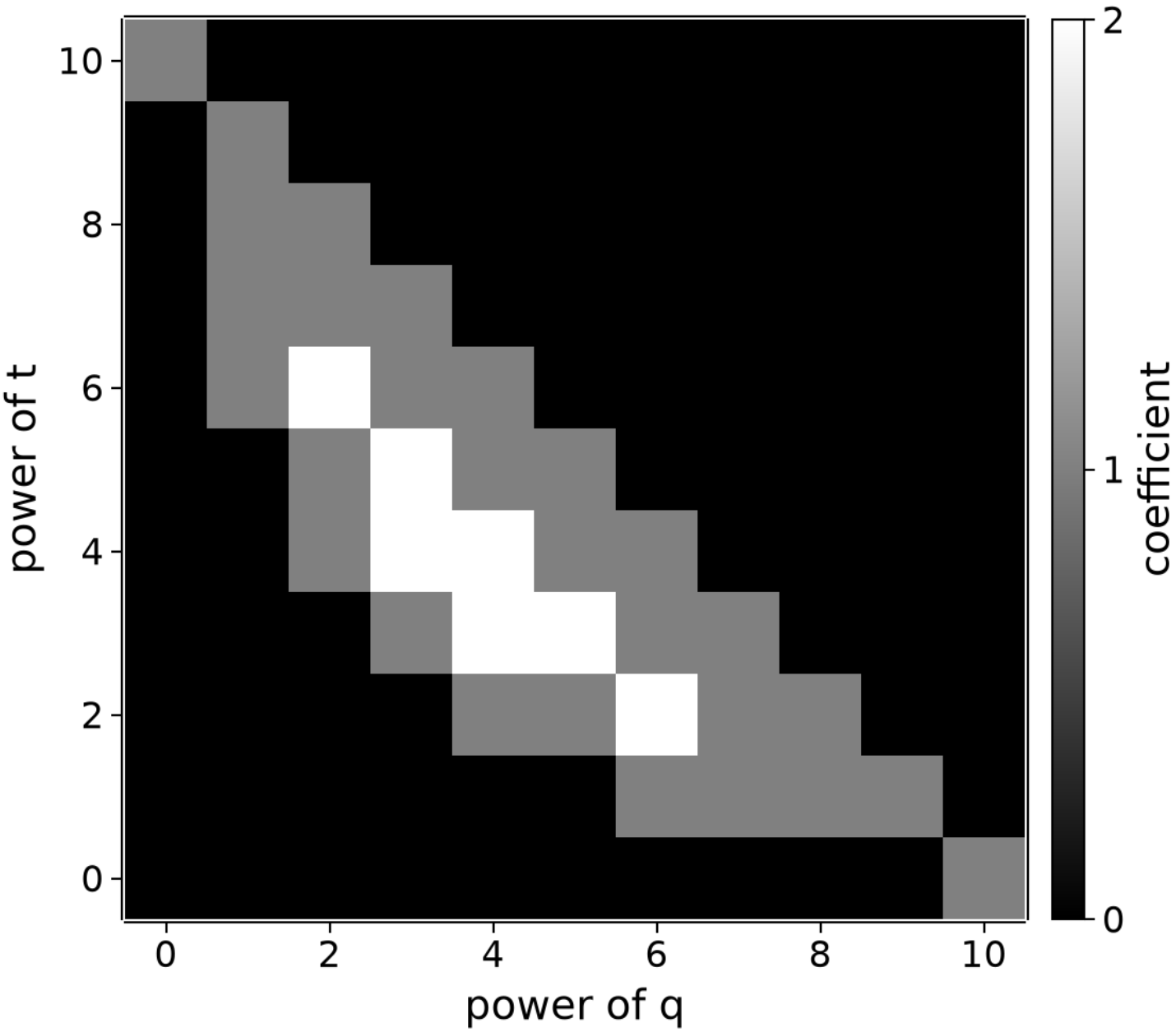
$$\begin{aligned} F(0,1,2) = {} & q^8 + q^7 t + q^6 t^2 + q^5 t^3 + q^4 t^4 + q^3 t^5 + q^2 t^6 + q t^7 + t^8 \\ & + q^6 t + q^5 t^2 + q^4 t^3 + q^3 t^4 + q^2 t^5 + q t^6 + q^5 t + 2q^4 t^2 \\ & + 2q^3 t^3 + 2q^2 t^4 + q t^5 - q^4 t - q^3 t^2 - q^2 t^3 - q t^4. \end{aligned}$$

But conjectured to be when $a_2 \geq \cdots \geq a_n$.

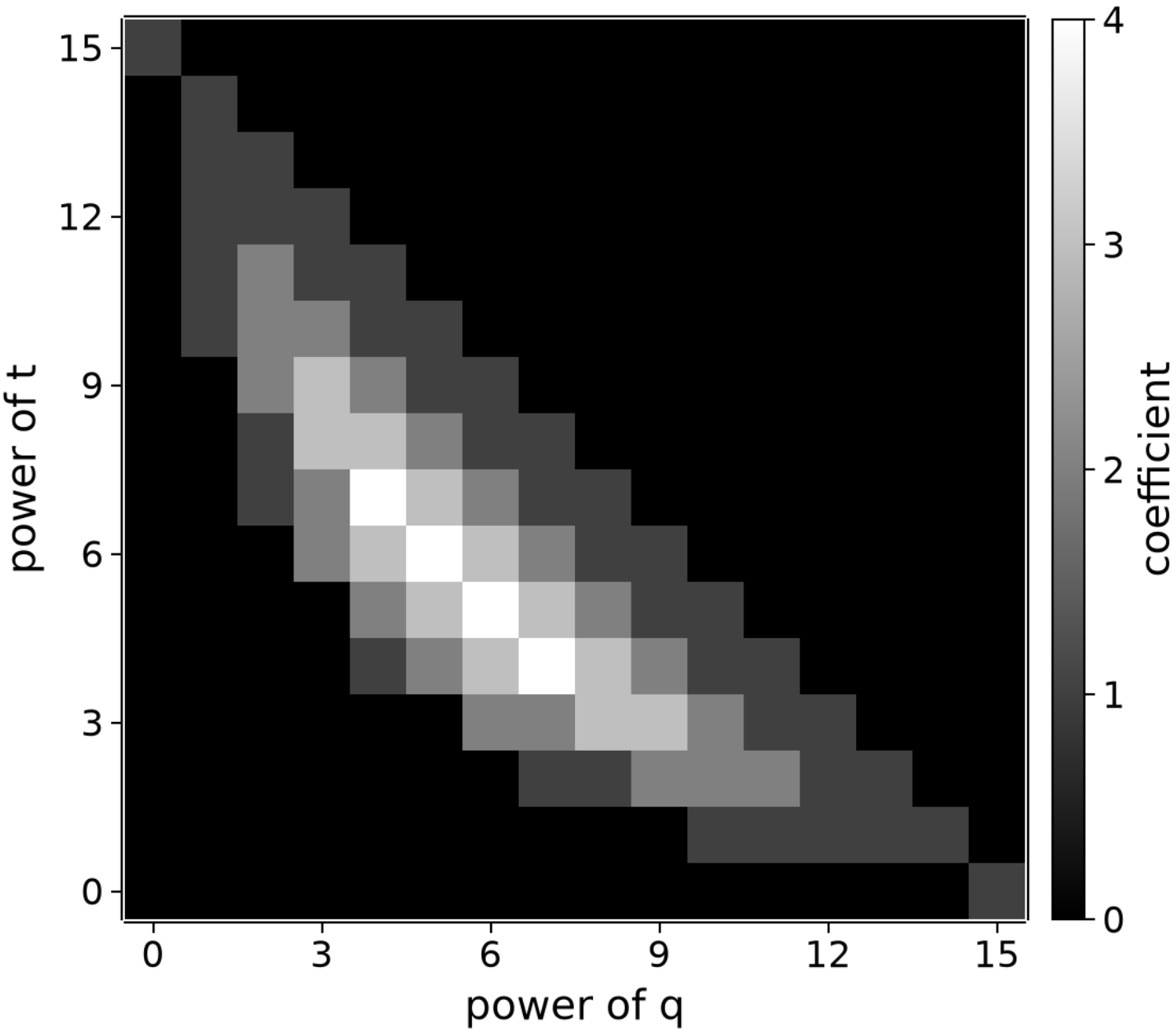
Coefficient grids

10	1									
9		1								
8		1	1							
7		1	1	1						
6		1	2	1	1					
5			1	**2**	1	1				
4			1	2	2	1	1			
3				1	2	2	1	1		
2					1	1	2	1	1	
1							1	1	1	1
0										1
t/q	0	1	2	**3**	4	5	6	7	8	9	10

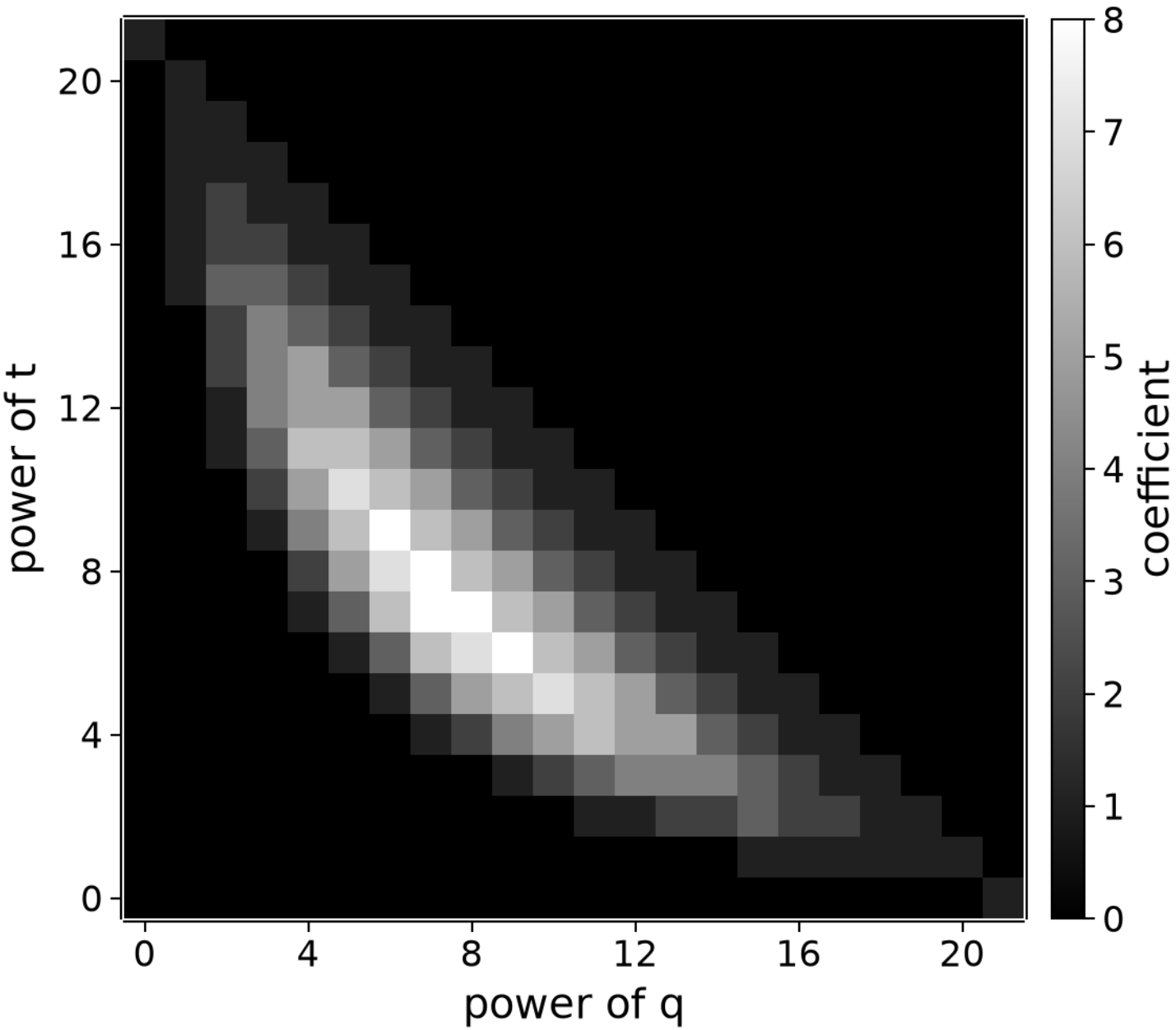
$$C_5(q,t) = F(1,1,1,1) = \cdots + 2q^3t^5 + \cdots$$



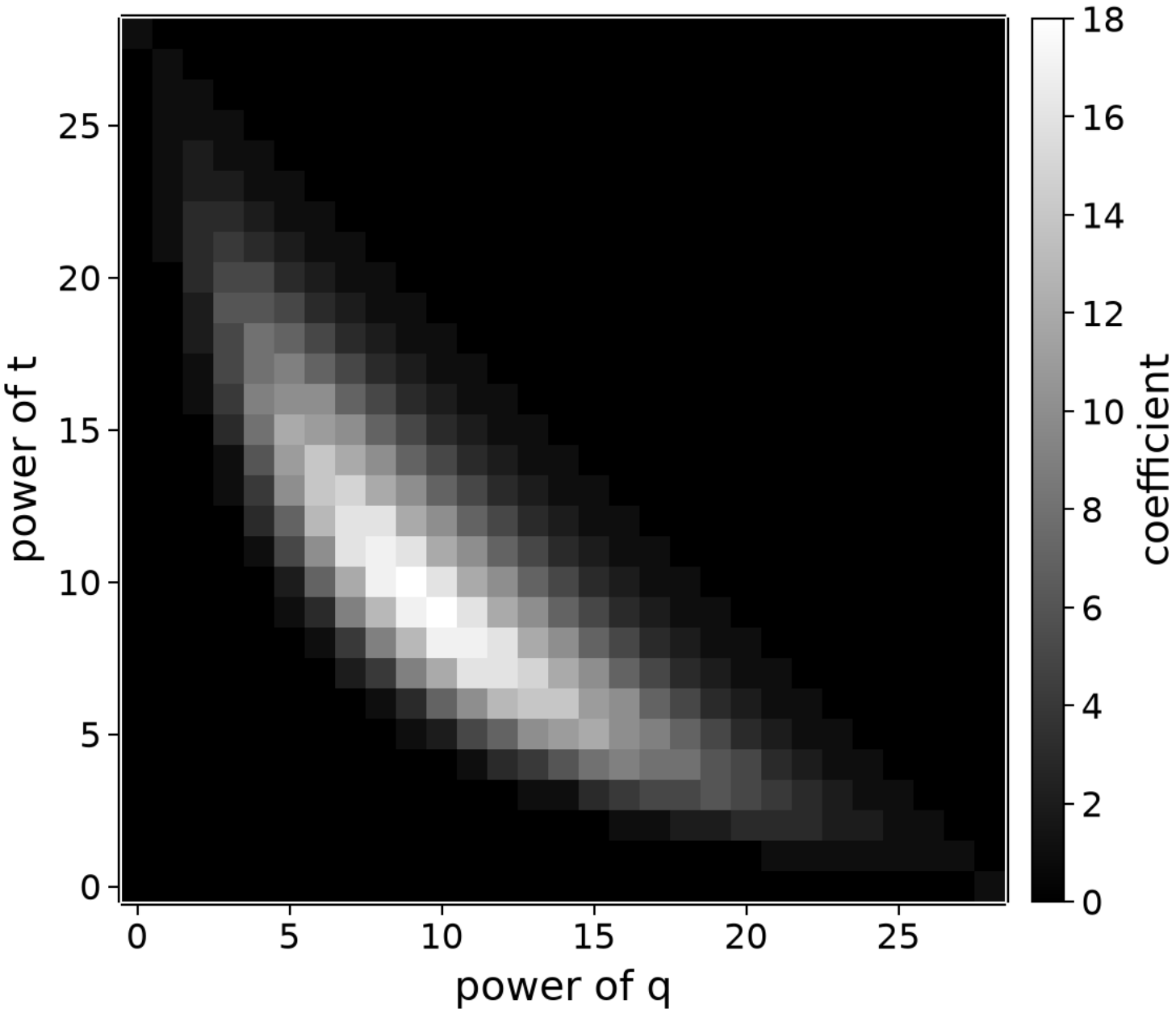
$$C_5(q,t) = F(1,1,1,1)$$



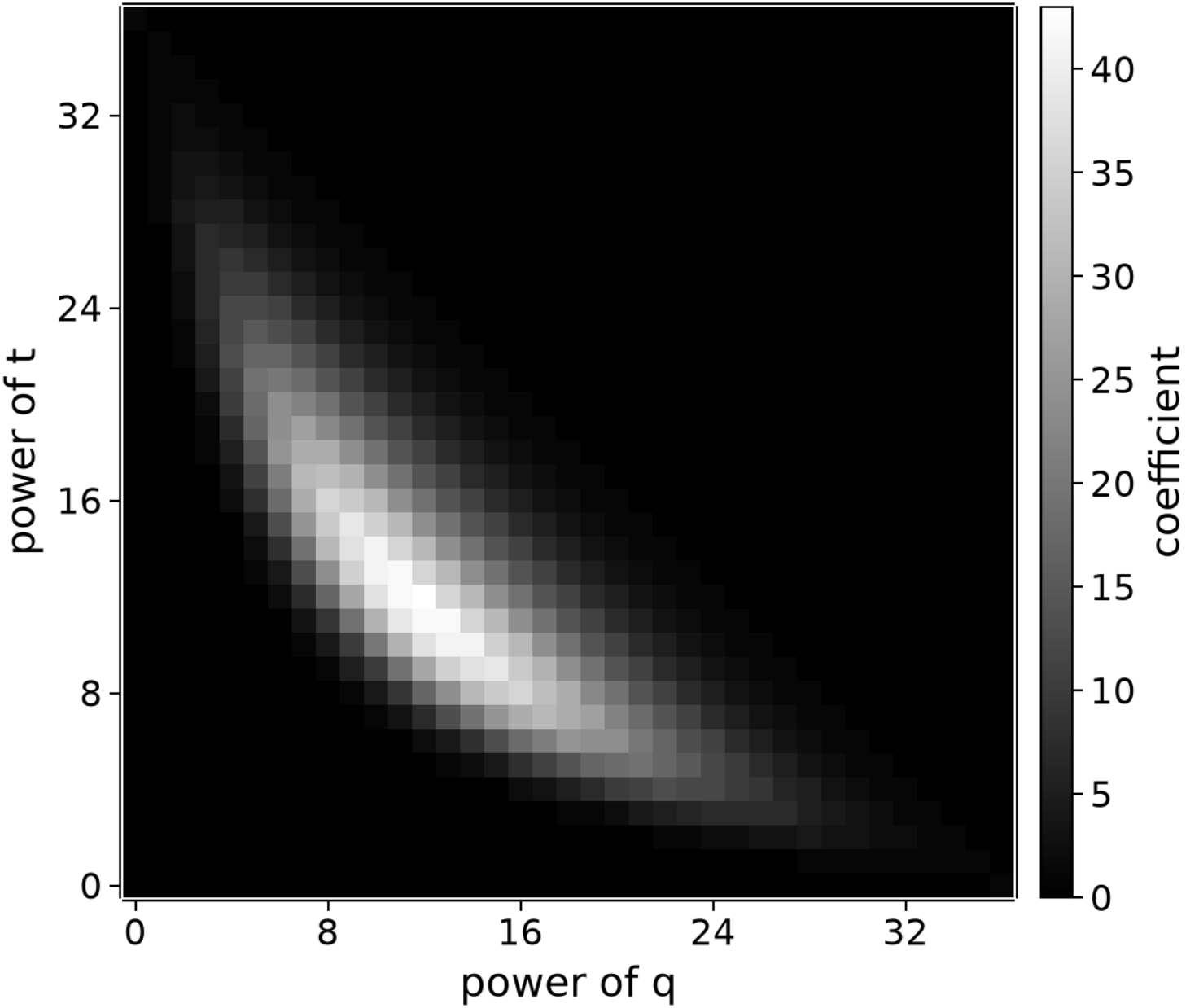
$$C_6(q,t) = F(1,1,1,1,1)$$



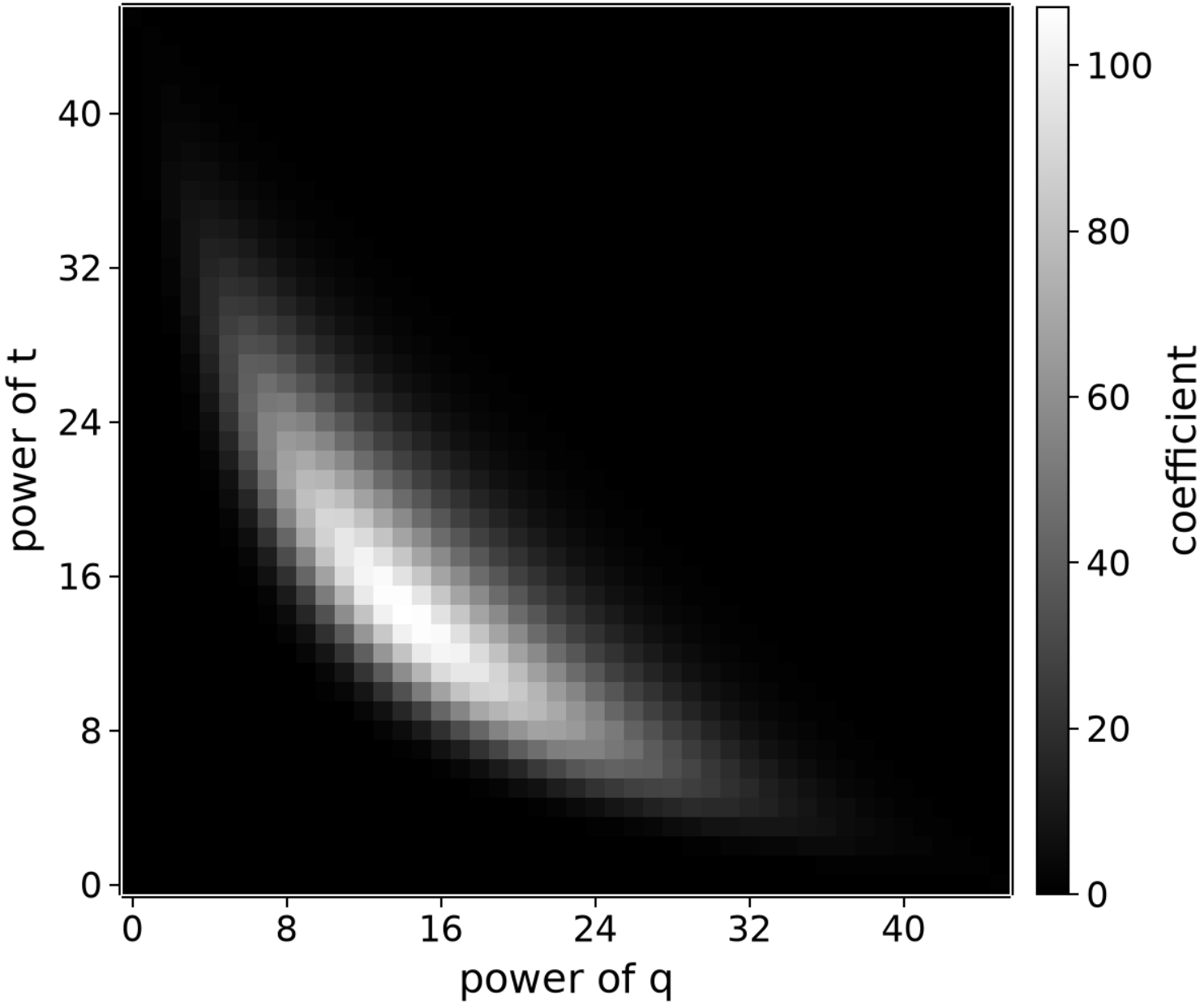
$$C_7(q,t) = F(1,1,1,1,1,1)$$



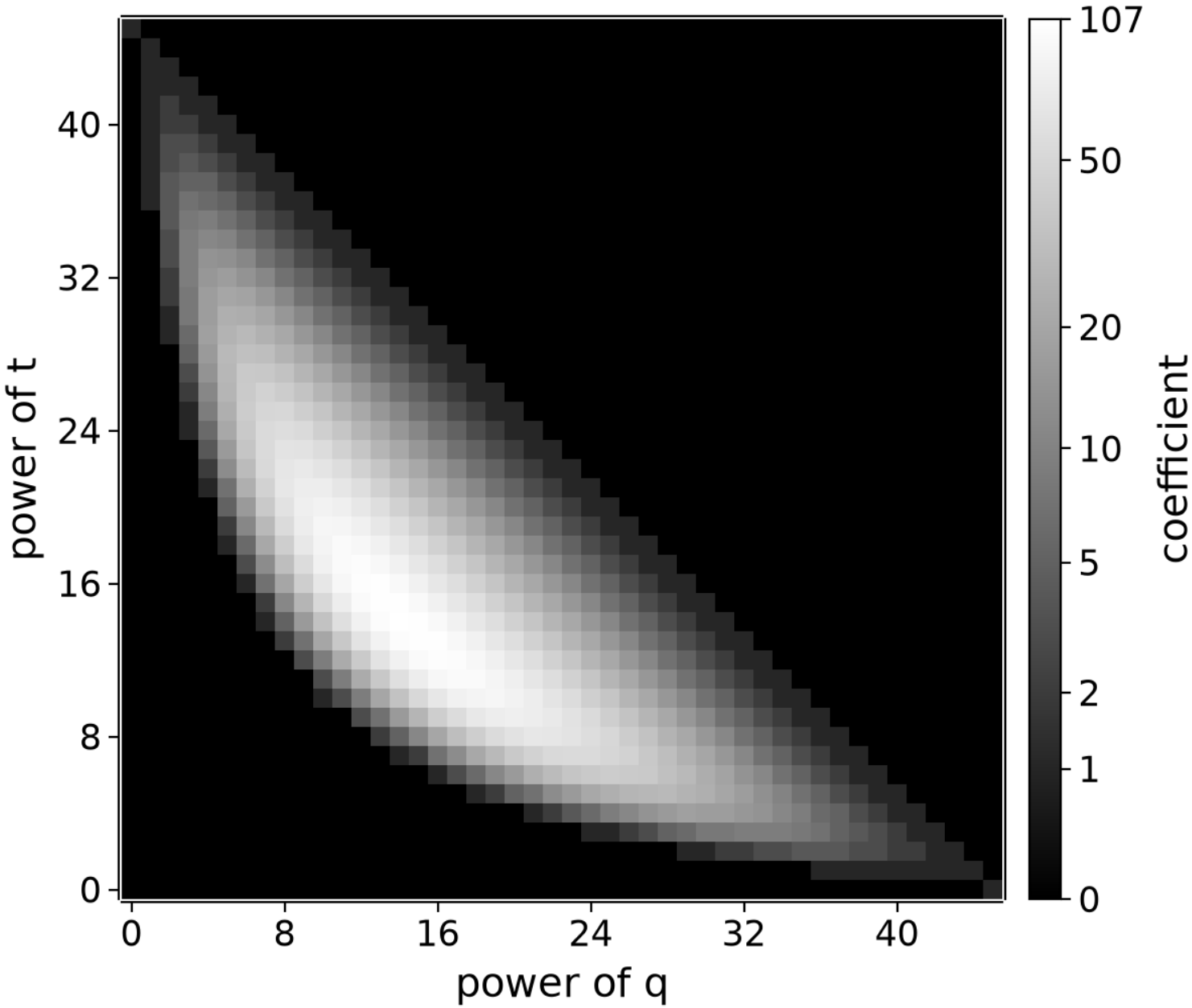
$$C_8(q,t) = F(1,1,1,1,1,1,1)$$



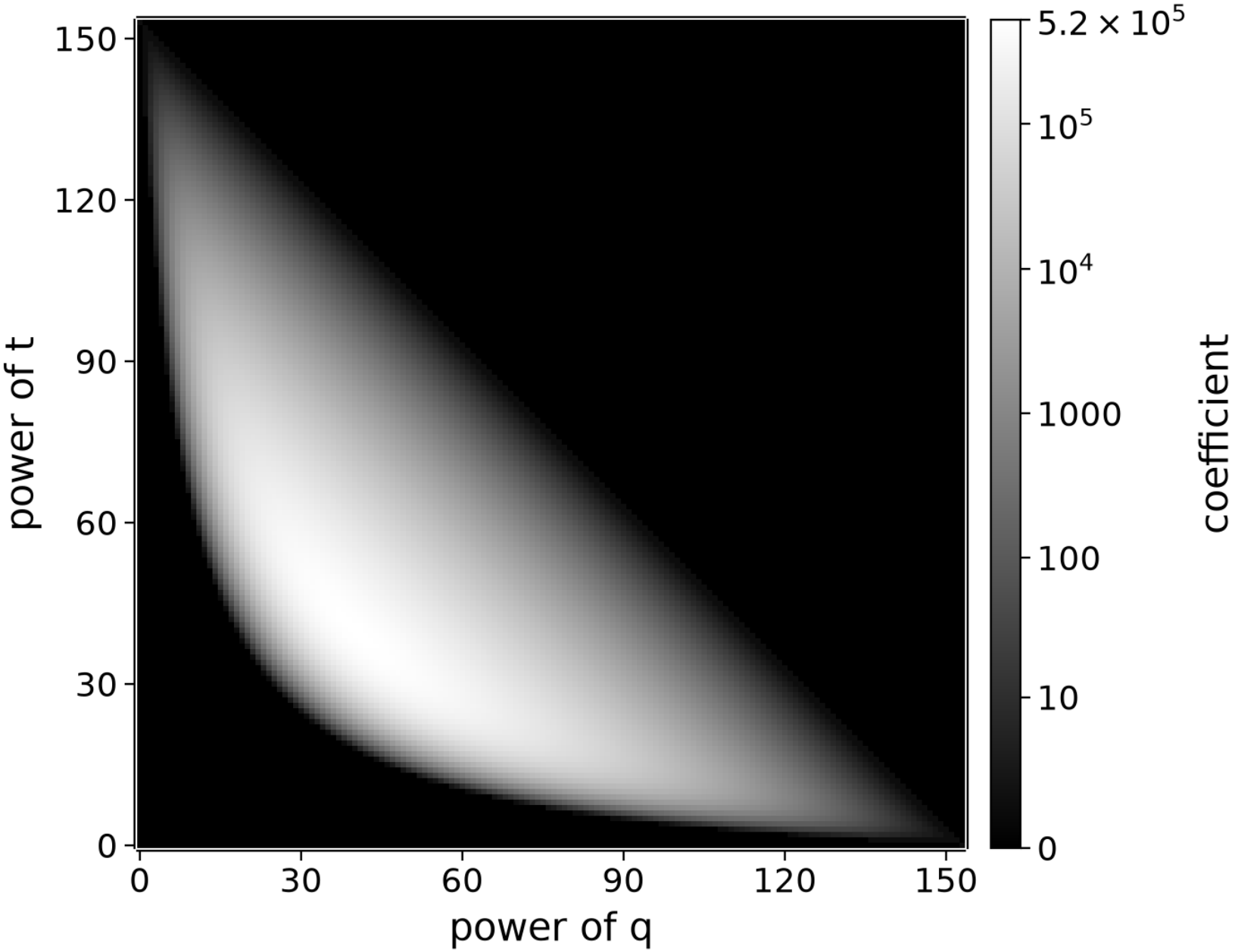
$$C_9(q,t) = F(1,1,1,1,1,1,1,1,1)$$



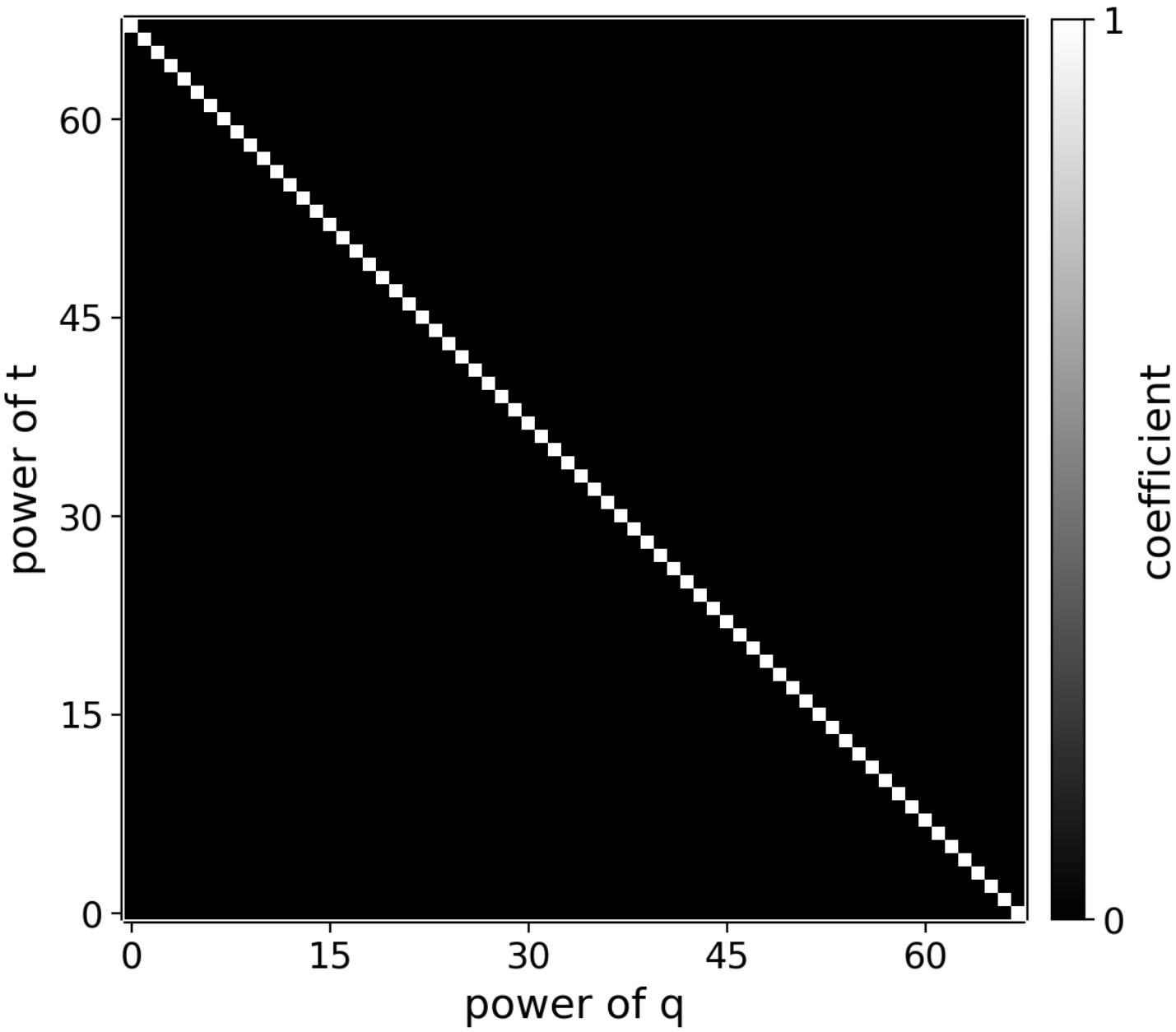
$$C_{10}(q,t) = F(1,1,1,1,1,1,1,1,1,1)$$



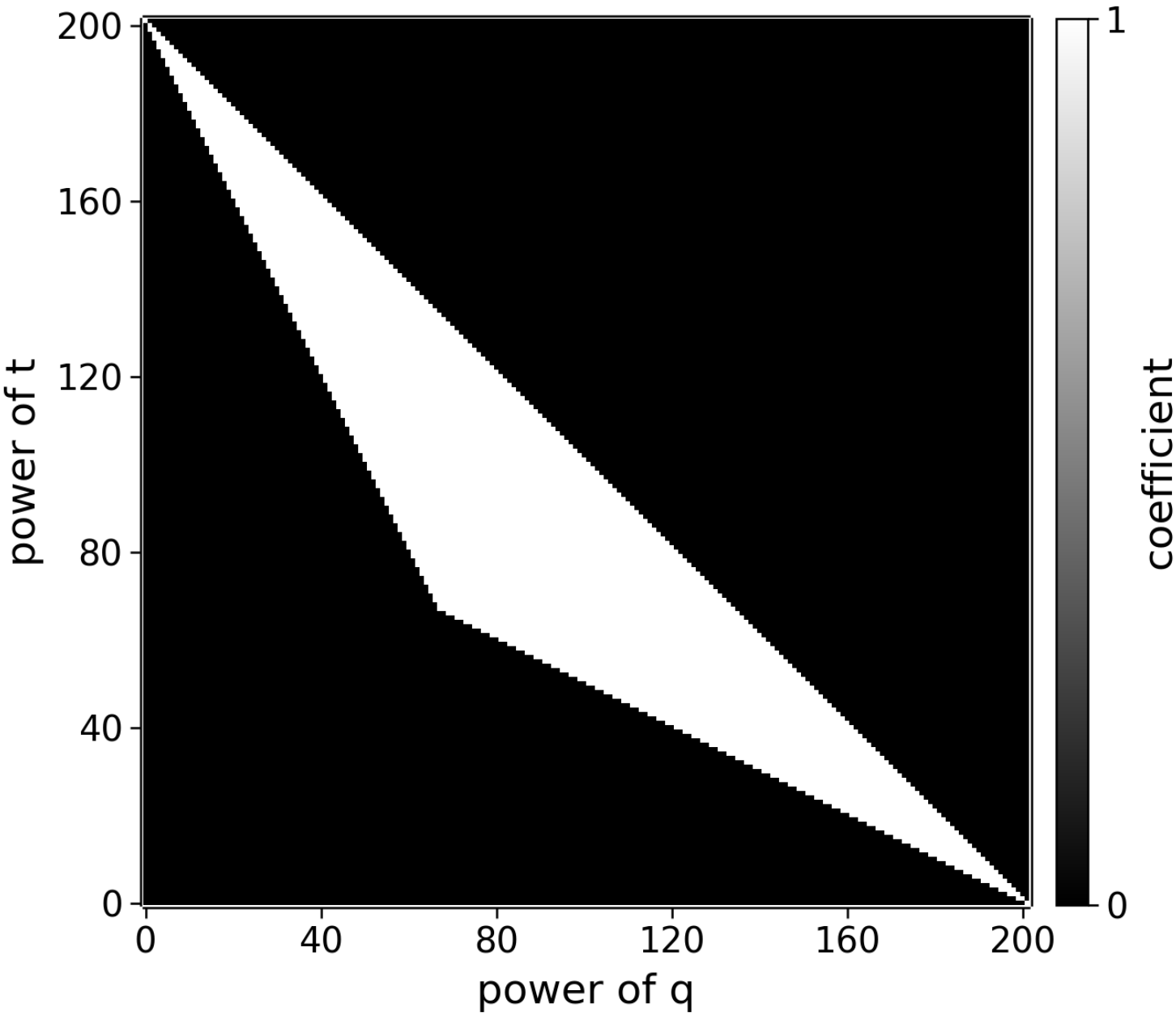
$$C_{10}(q,t) = F(1,1,1,1,1,1,1,1,1)$$



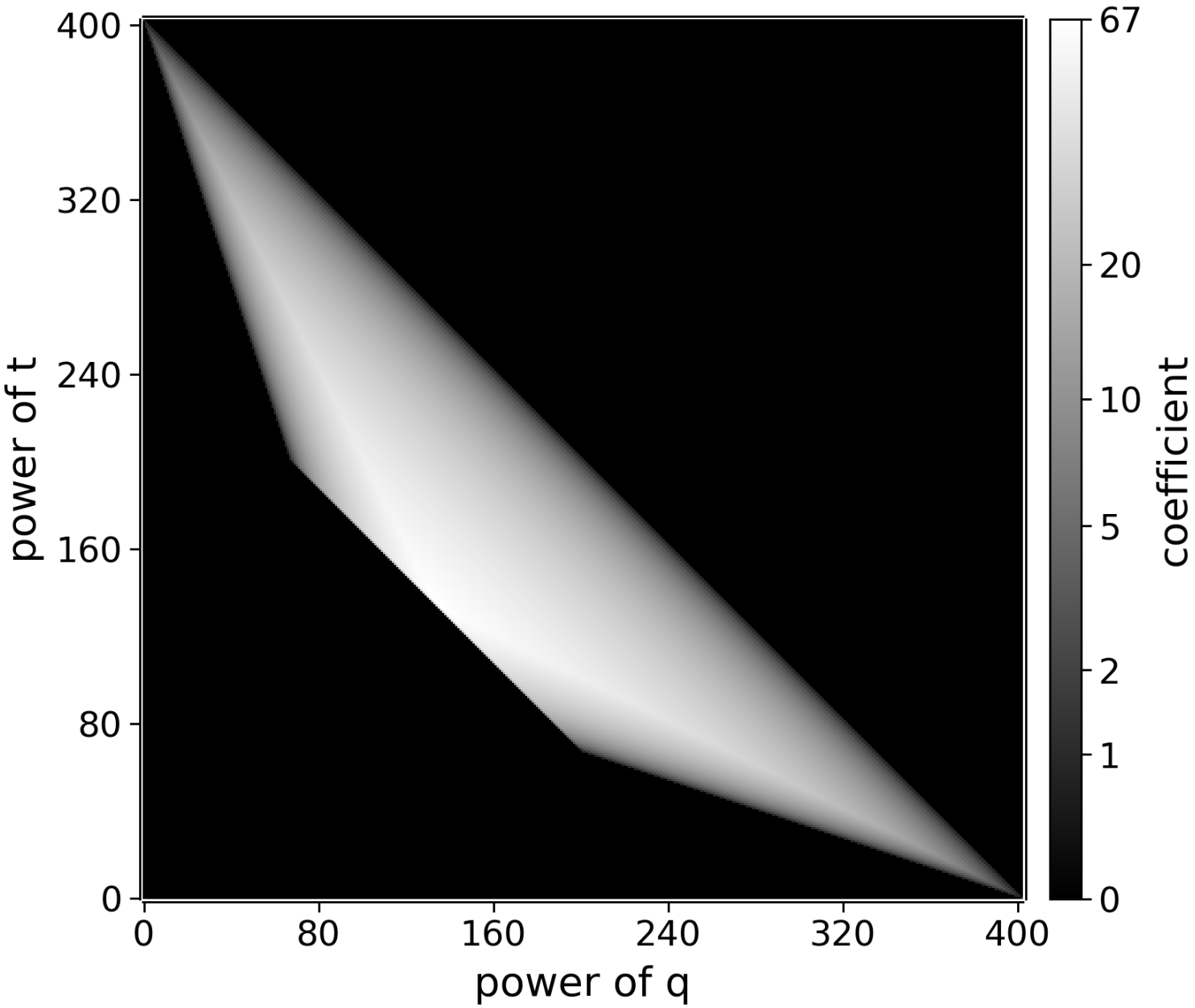
$$C_{18}(q,t) = F(1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1)$$



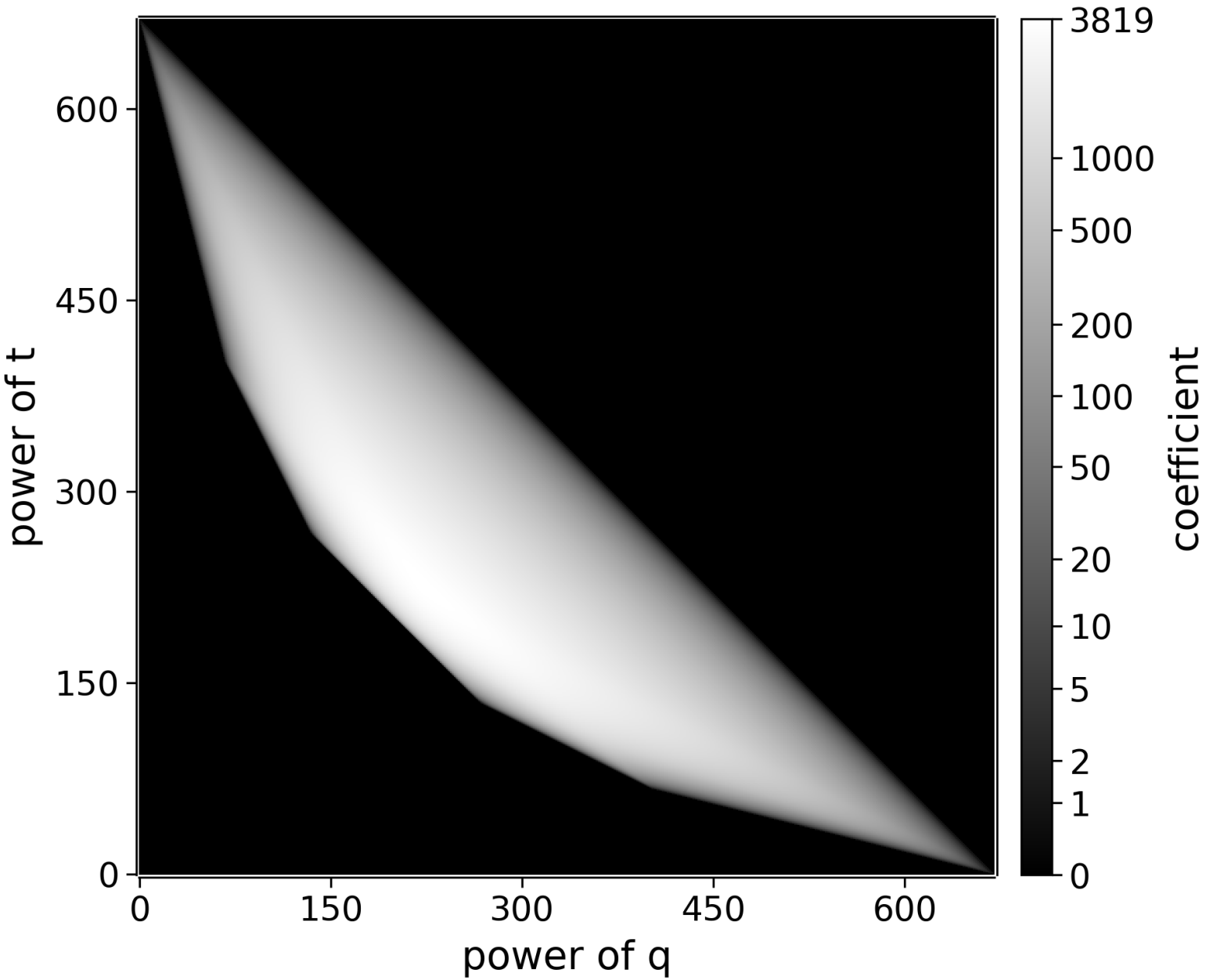
$F(67)$


200
160
120
80
40
0
power of t
0
40
80
120
160
200
power of q
1
0
coefficient

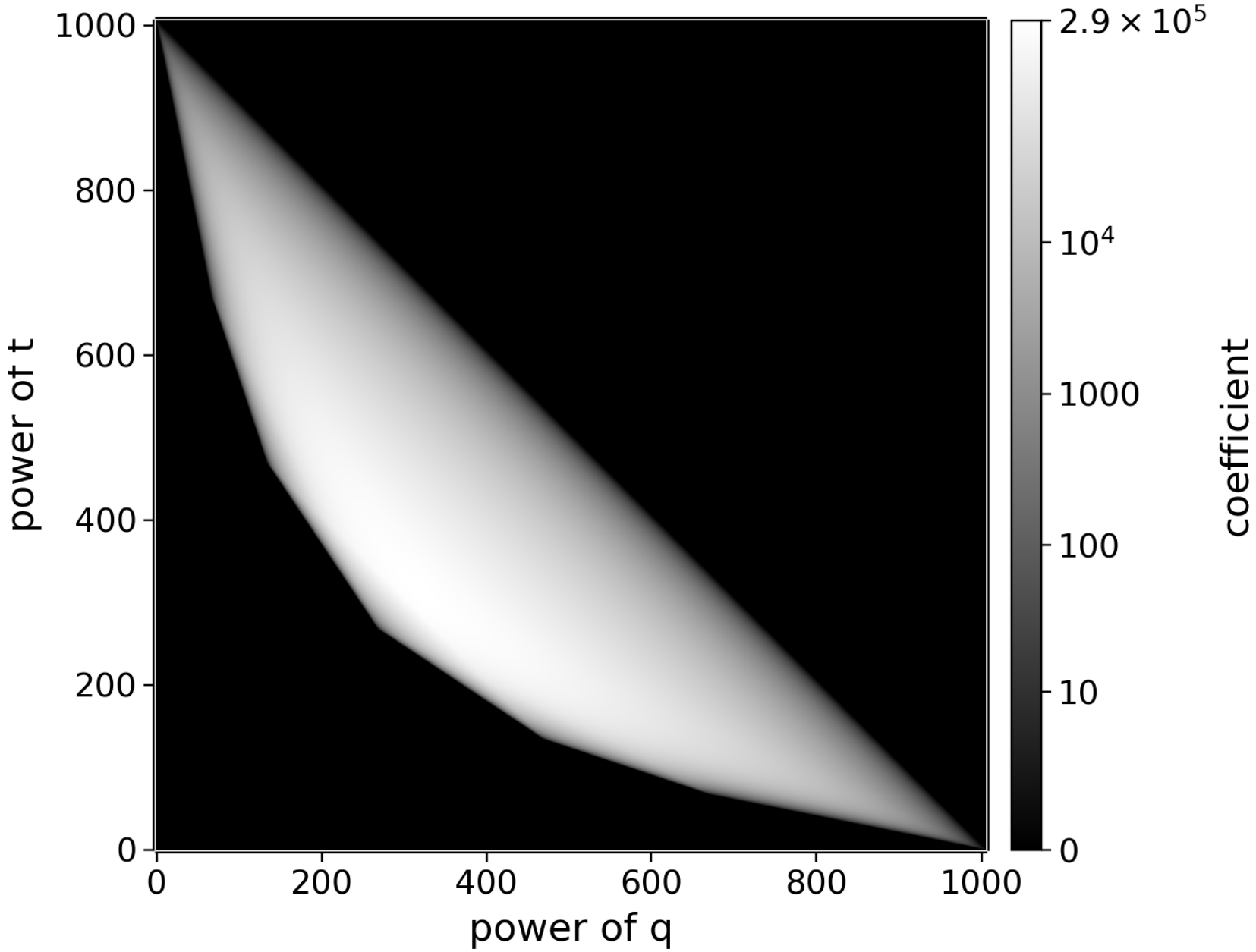
$F(67, 67)$


400
320
240
160
80
0
power of t
0
80
160
240
320
400
power of q
67
20
10
5
2
1
0
coefficient

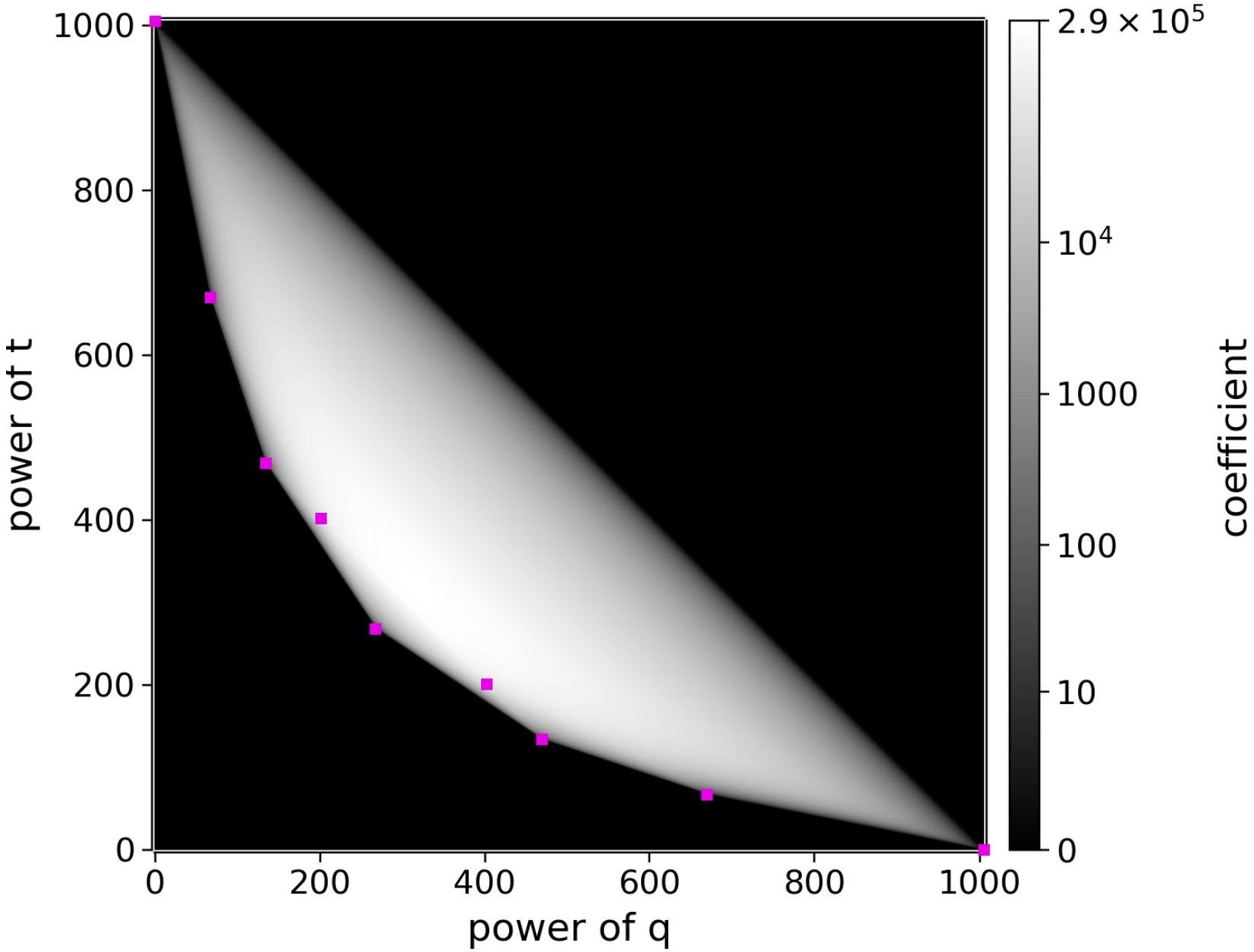
$F(67, 67, 67)$


power of t
power of q
coefficient
0
150
300
450
600
3819
1000
500
200
100
50
20
10
5
2
1
0

$F(67, 67, 67, 67)$



$F(67, 67, 67, 67, 67)$



$F(67, 67, 67, 67, 67)$ with tableau points.

Convex hulls

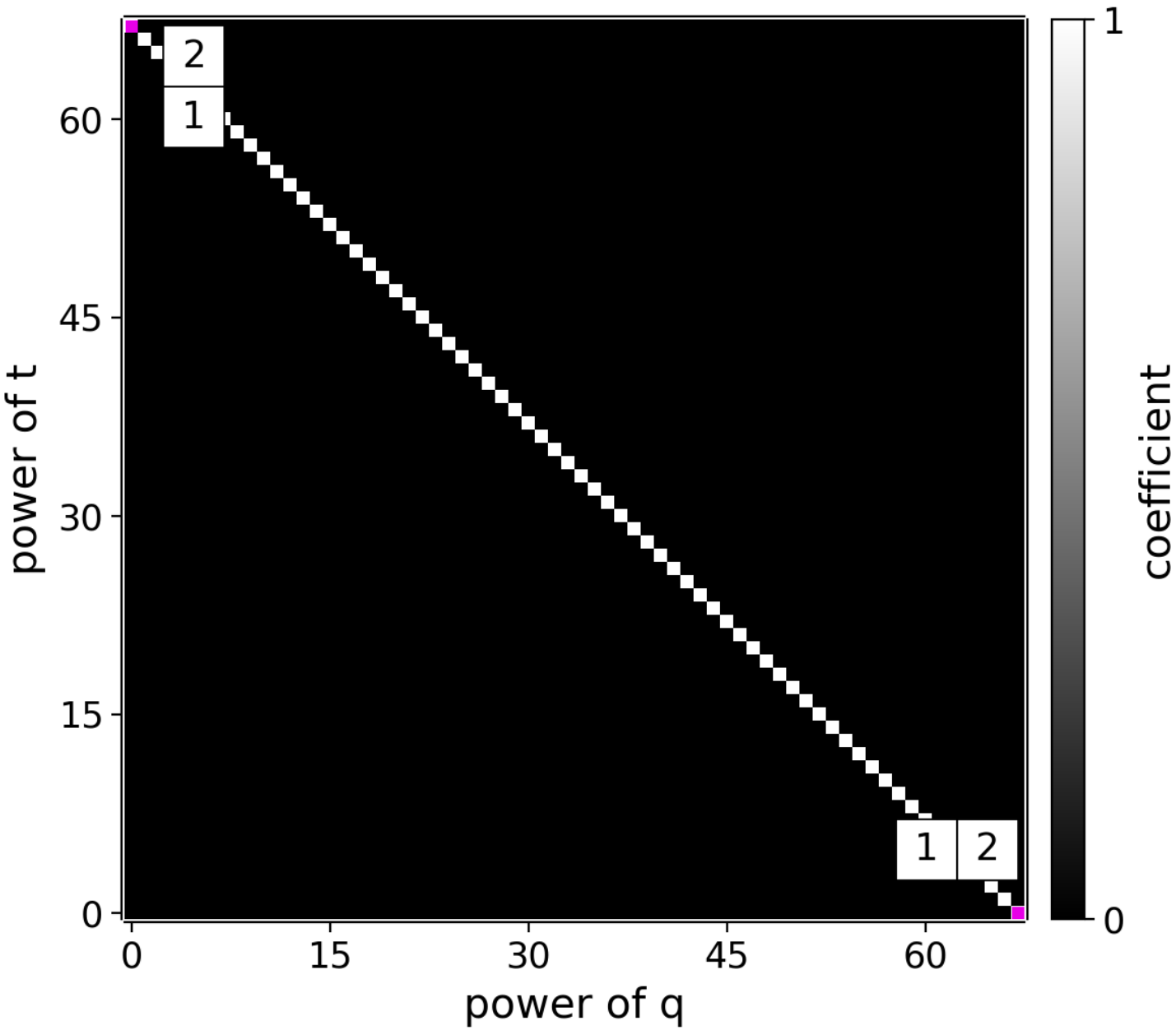
Recall that

$$F(a_2, \ldots, a_n) = \sum_T z_2^{a_2} \ldots z_n^{a_n} \operatorname{wt}(T).$$

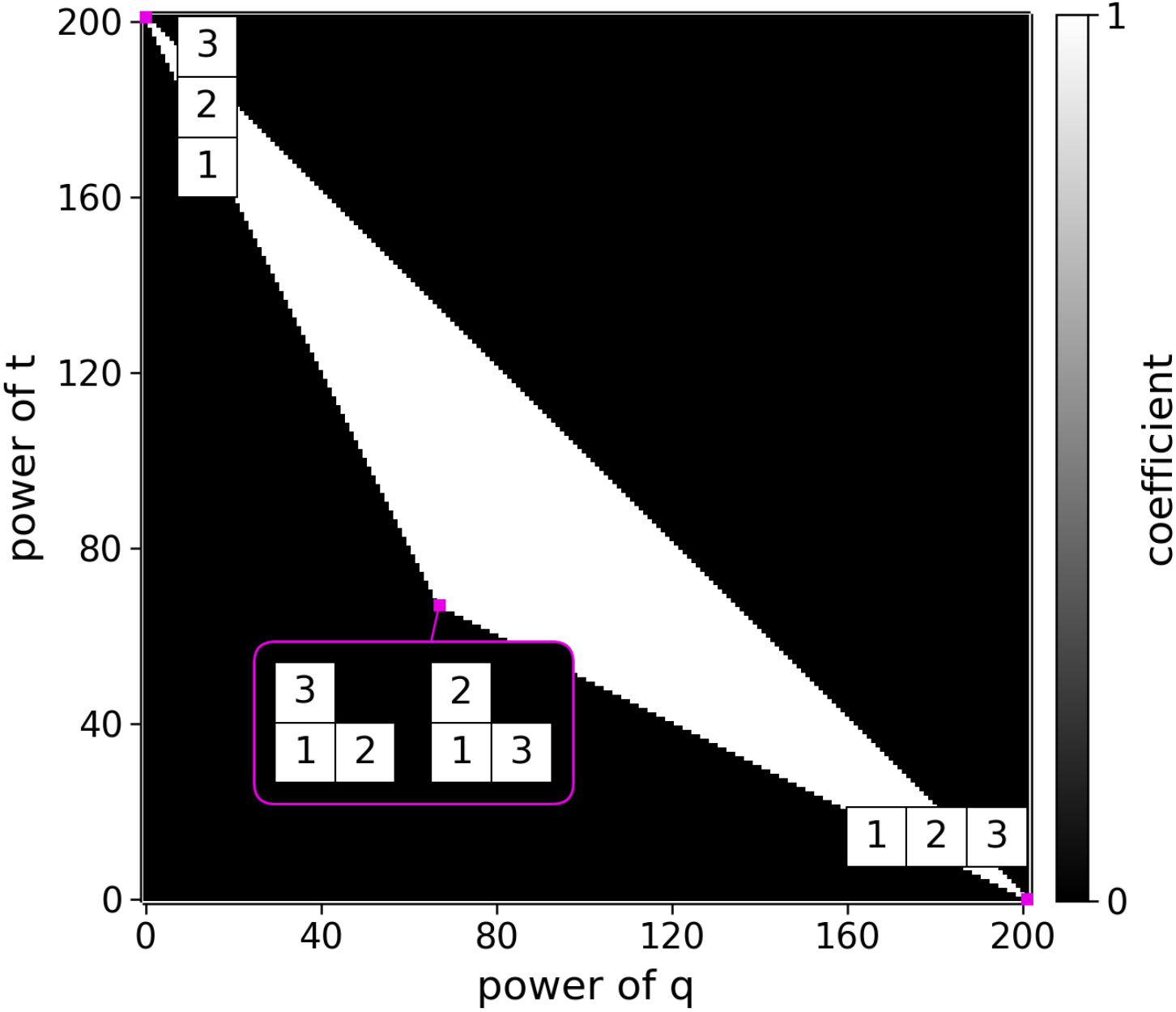
Plot the coefficients $z_2^{a_2} \ldots z_n^{a_n}$ for all n-cell T; these are tableau points.

Theorem (L. 2026)

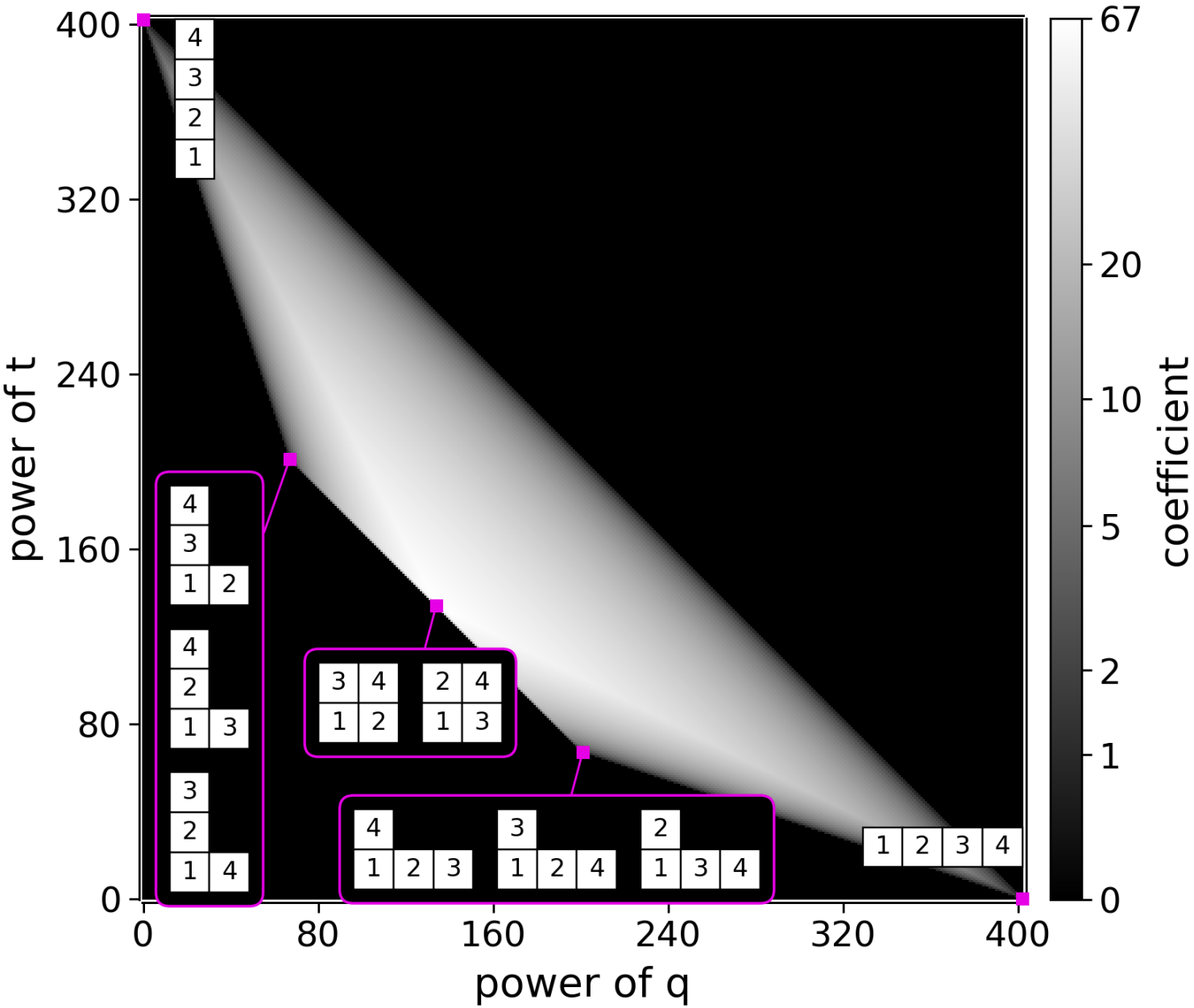
Coefficients outside the convex hull of the tableau points are zero.



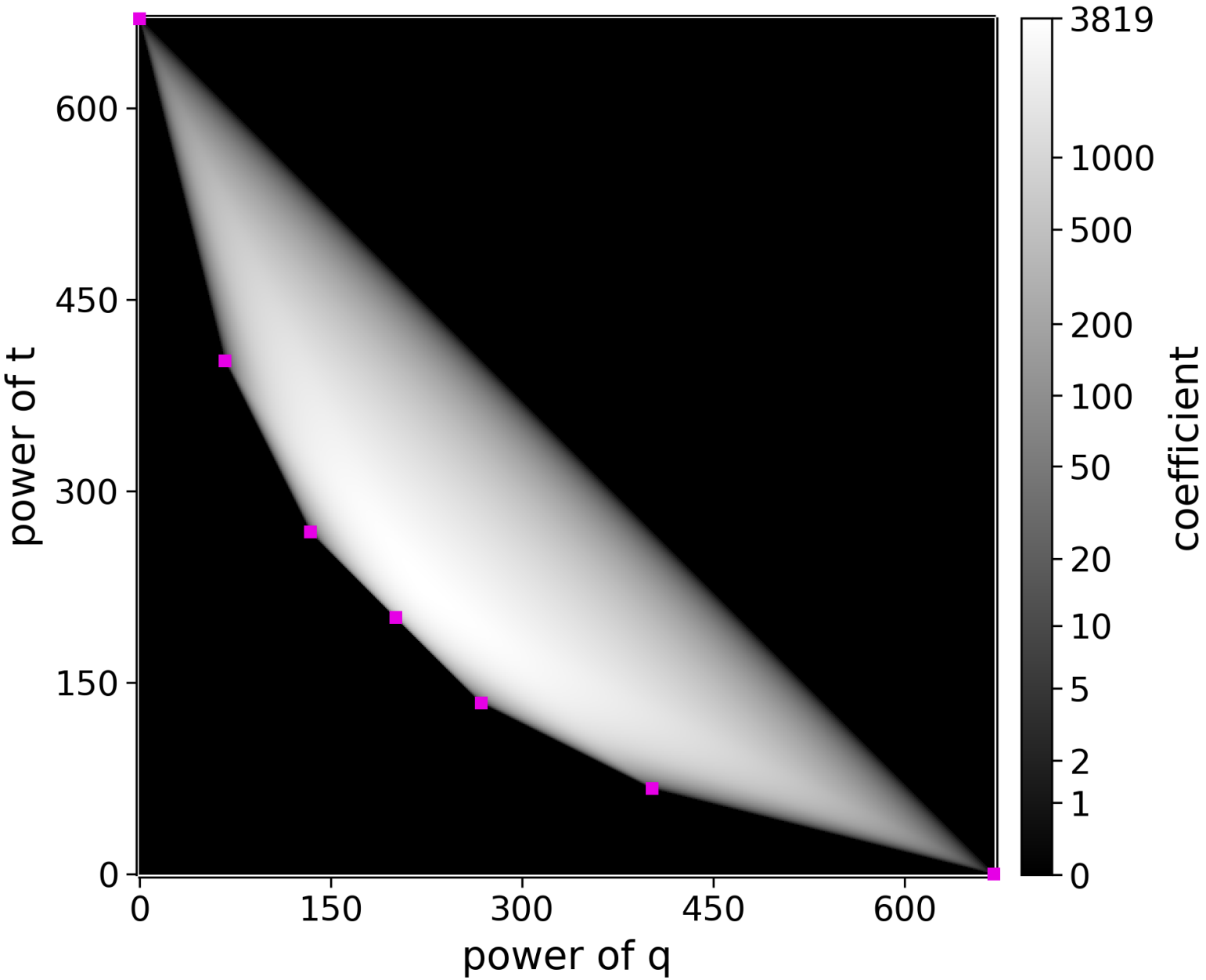
$F(67)$ with tableau points z_2^{67}



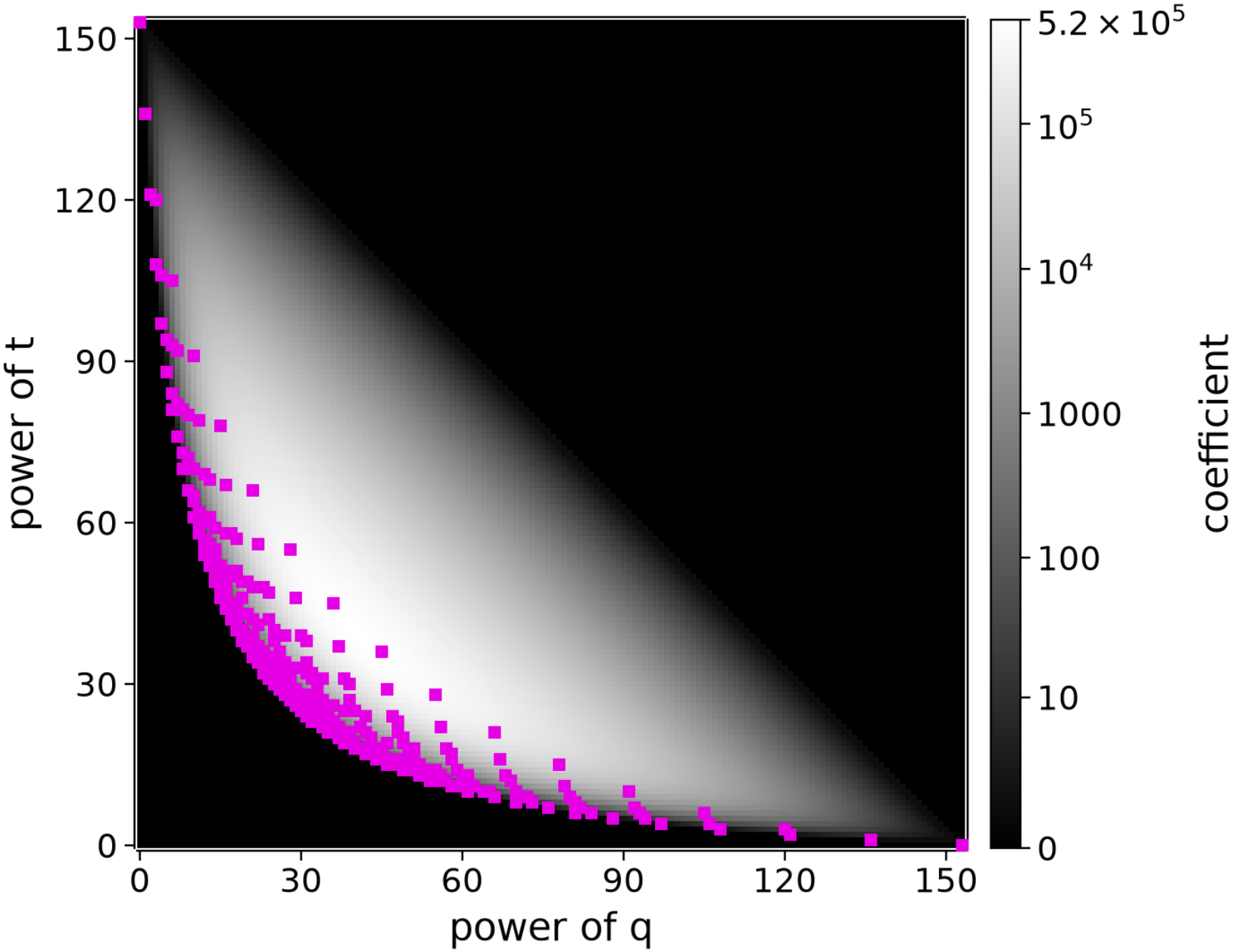
$F(67, 67)$ with tableau points $z_2^{67} z_3^{67}$



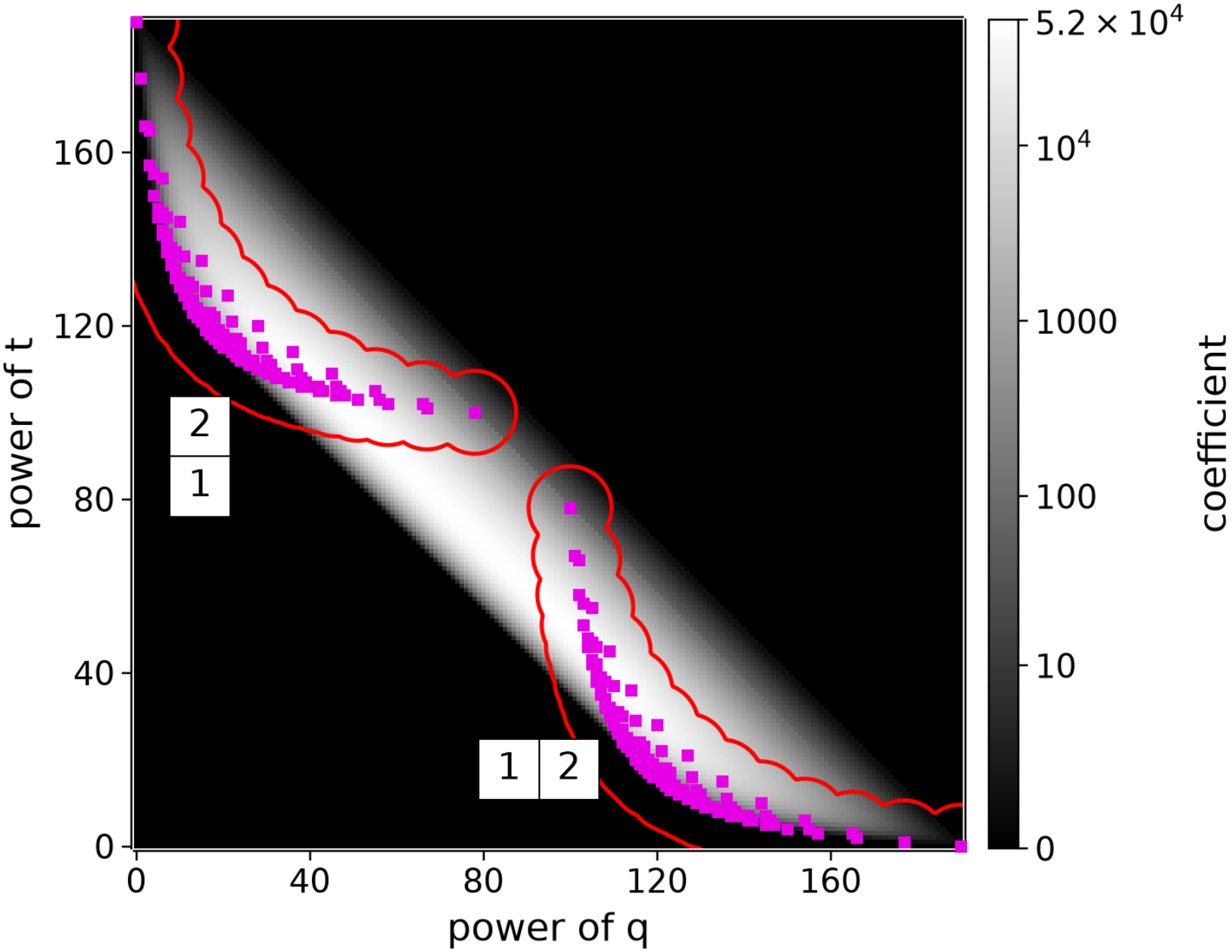
$F(67, 67, 67)$ with tableau points $z_2^{67} z_3^{67} z_4^{67}$



$F(67, 67, 67, 67)$ with tableau points $z_2^{67} z_3^{67} z_4^{67} z_5^{67}$



$C_{18}(q,t)$ with tableau points $z_2 \ldots z_{18}$.



$F(100, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1)$ with tableau points $z_2^{100} z_3 \dots z_{14}$

Convex hulls

For all T let $z_2^{a_2} \dots z_n^{a_n} = q^{i_T} t^{j_T}$. Then (i_T, j_T) is *tableau point* of T.

Theorem (L. 2026)

Coefficients outside the convex hull of the tableau points are zero.

Question

What is the shape of the convex hull of the tableau points?

Convex hulls

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Theorem (L. 2026)

As $n \to \infty$, the convex hull when $a_2 = \cdots = a_n = 1$ looks like the hyperbola $ij = \frac{2}{9}n^3$.

Centroids

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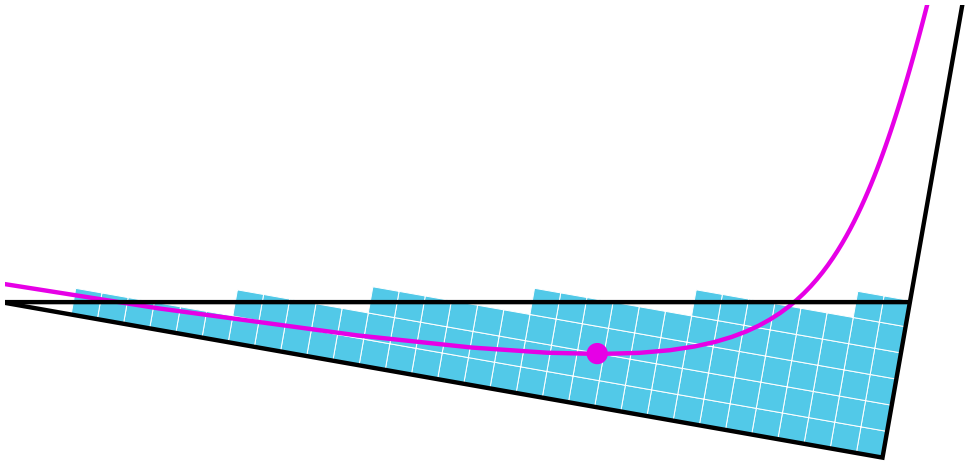
When $a_2 = \cdots = a_n = 1$, tableau point associated to T is n times centroid of shape of T.

$$T = \begin{array}{|c|c|c|c|c|c|}\hline 7 & & & & & \\ \hline 6 & 8 & 10 & & & \\ \hline 3 & 5 & 9 & 13 & & \\ \hline 1 & 2 & 4 & 11 & 12 & 14 \\ \hline \end{array}$$

$z_2 z_3 \dots z_n$: product of

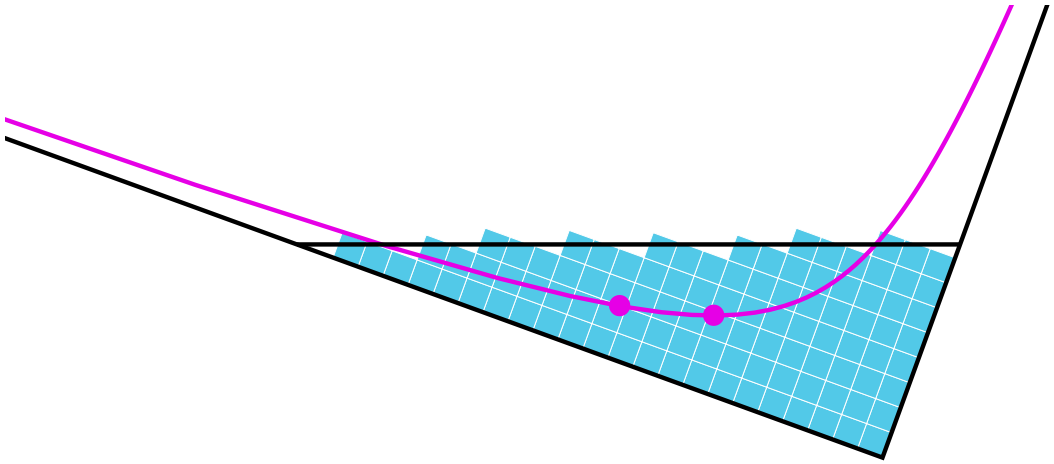
t^3					
t^2	qt^2	q^2t^2			
t	qt	q^2t	q^3t		
1	q	q^2	q^3	q^4	q^5

Liquid partitions



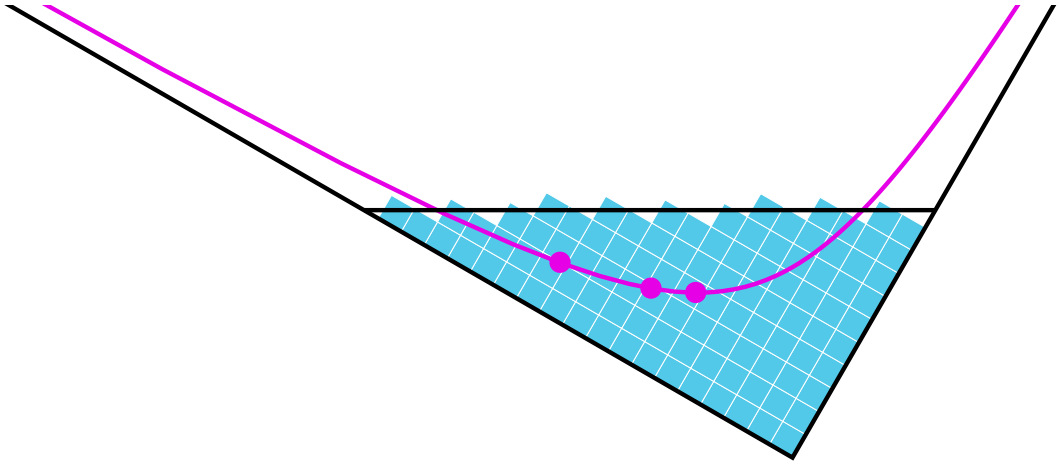
- Centroid smallest in some direction implies triangular partition
- As direction varies, locus of centroids traces hyperbola

Liquid partitions



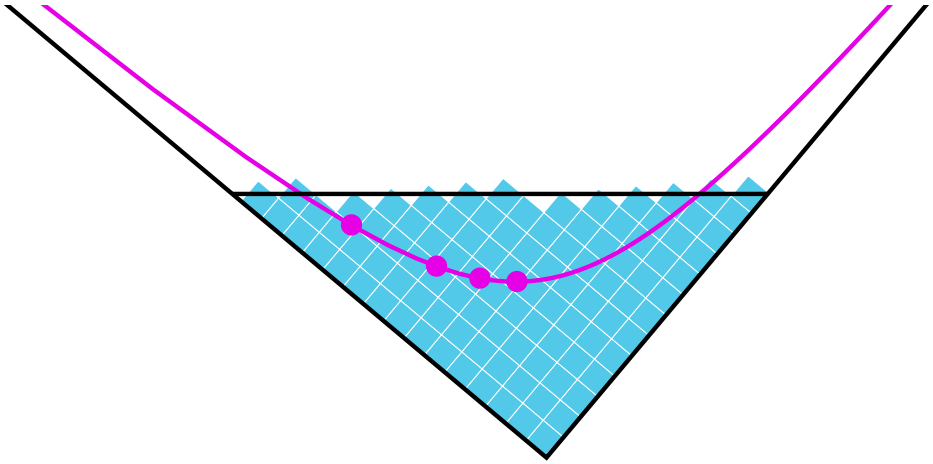
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Liquid partitions



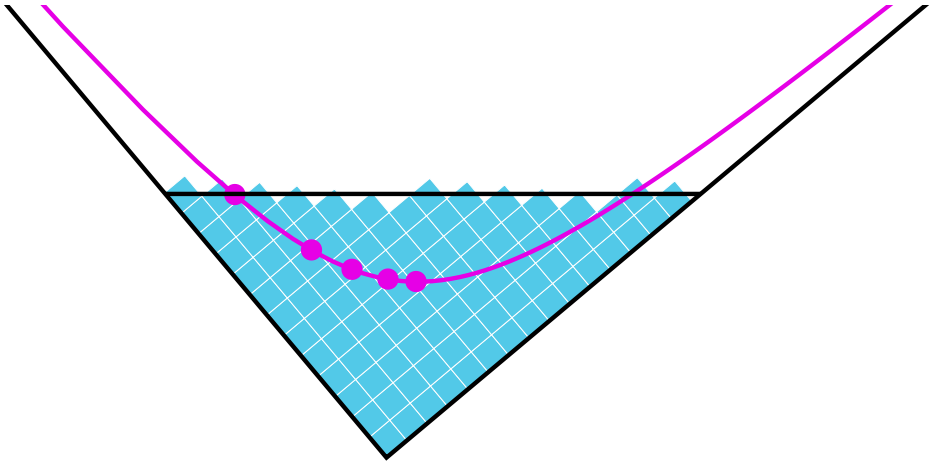
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Liquid partitions



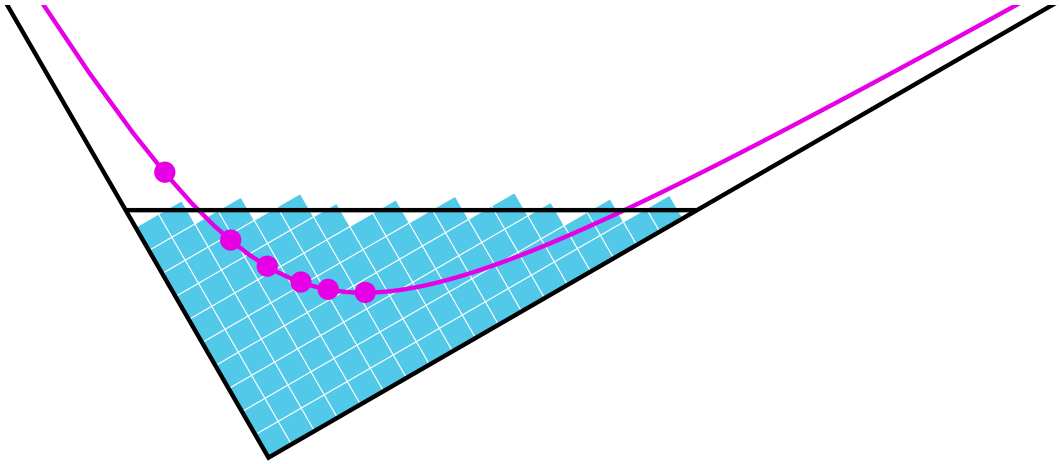
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Liquid partitions



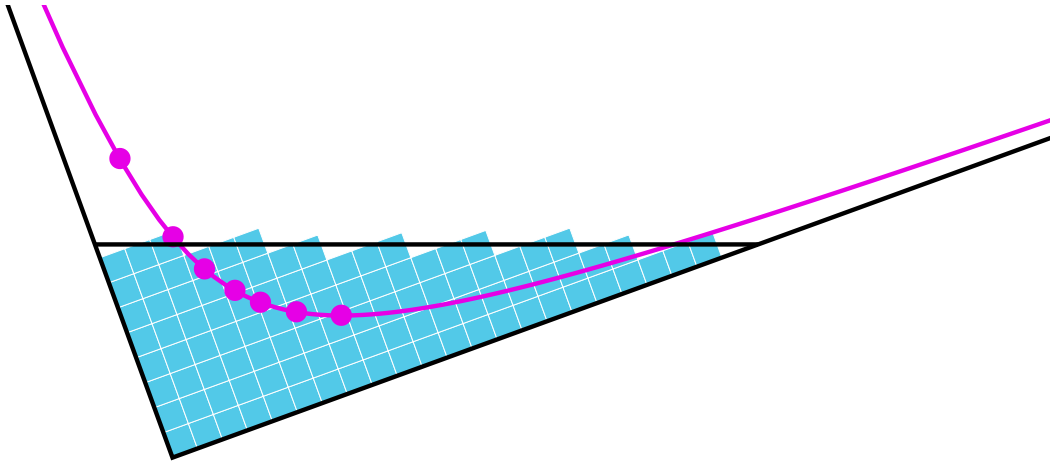
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Liquid partitions



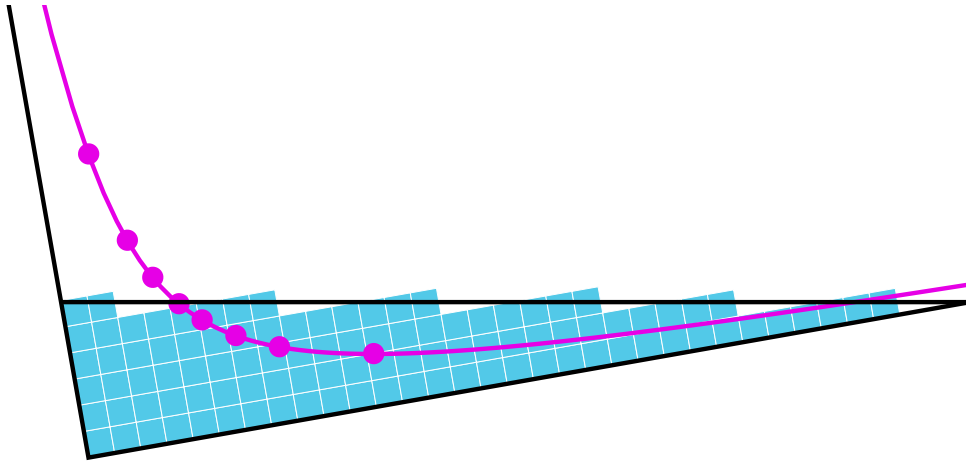
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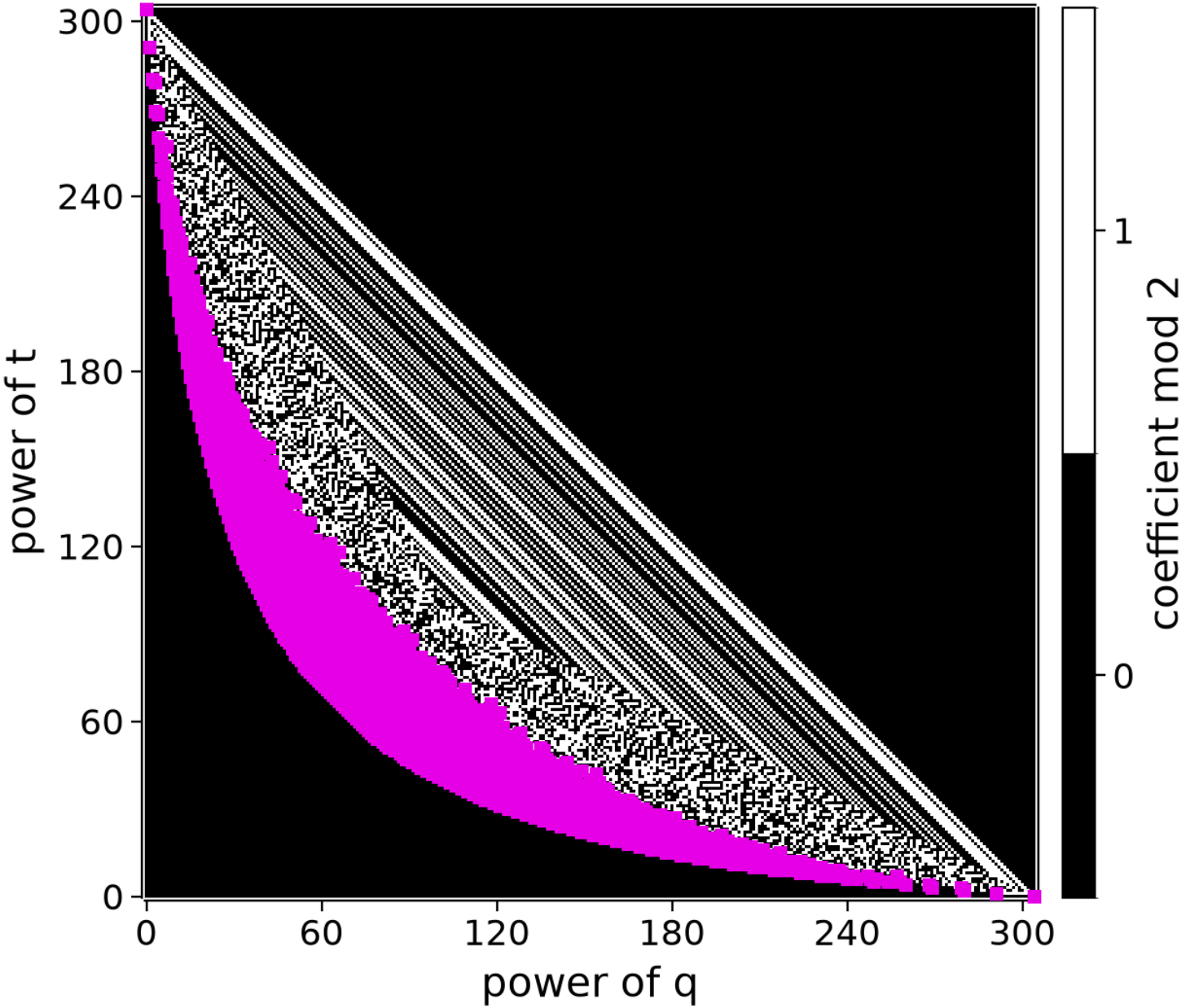


- Centroid smallest in some direction implies triangular partition
- As direction varies, locus of centroids traces hyperbola

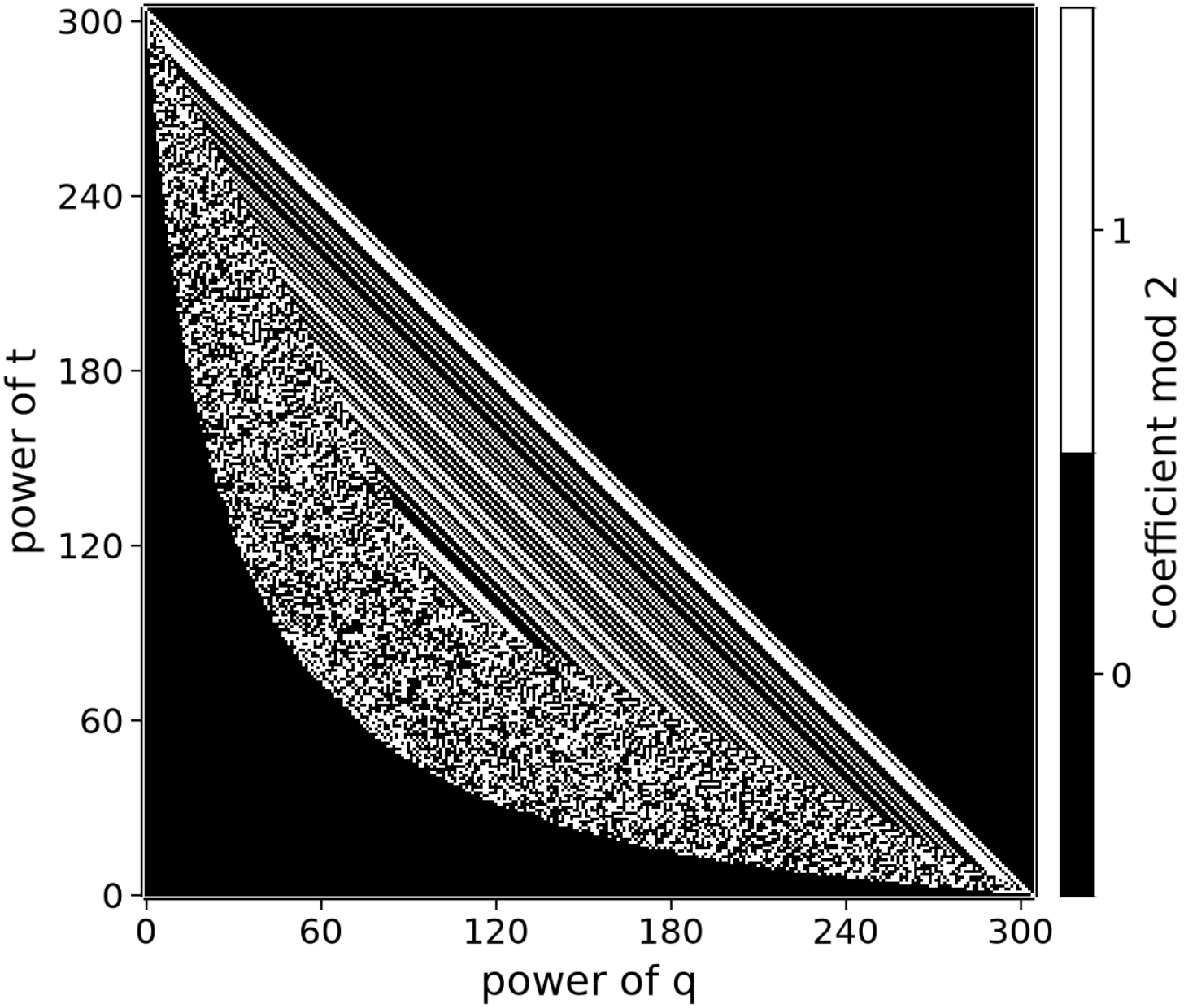
Liquid partitions



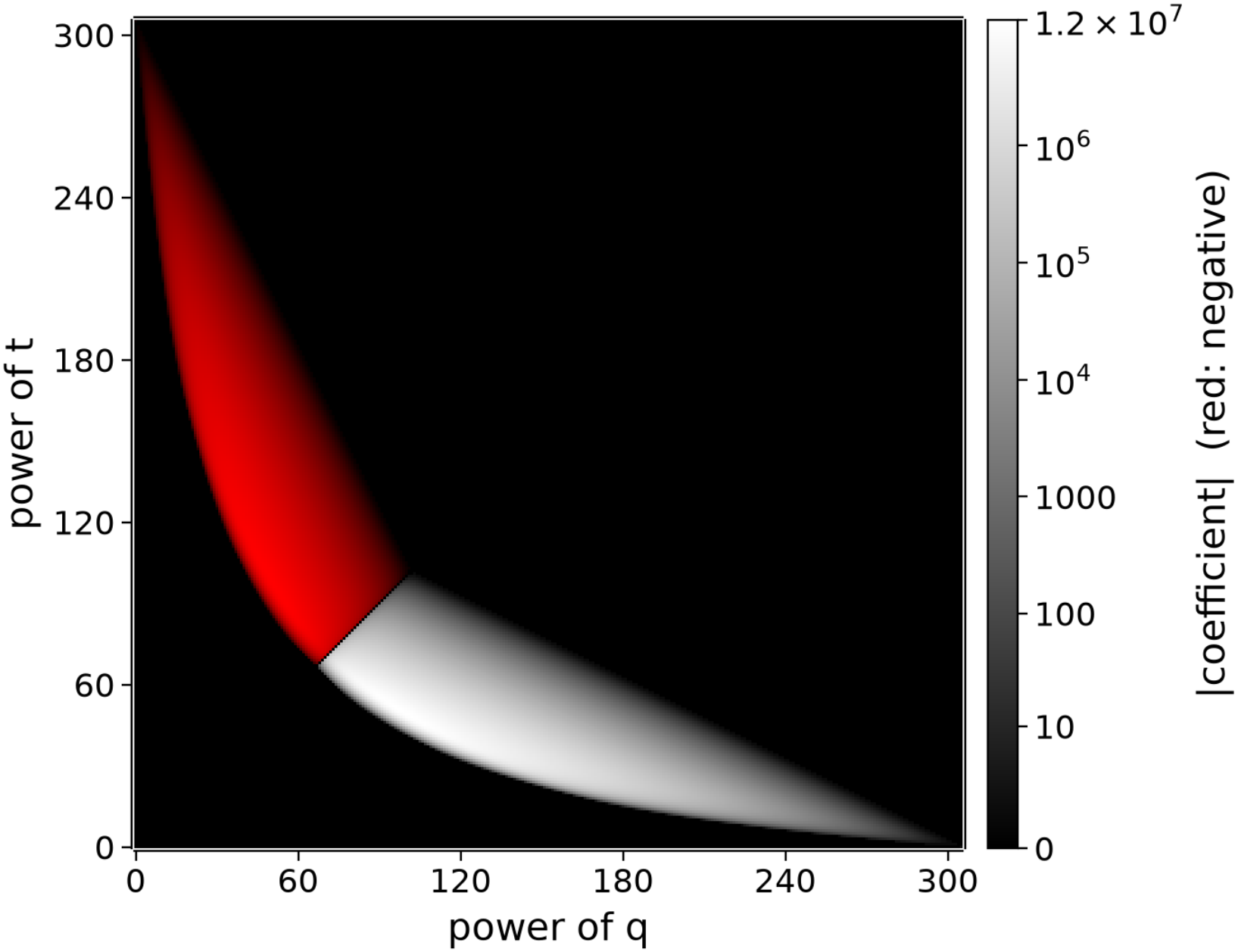
- Centroid smallest in some direction implies triangular partition
- As direction varies, locus of centroids traces hyperbola



$F(7,7,7,6,5,5,4,4,4,4,2,1,1) \bmod 2$



$F(7,7,7,6,5,5,4,4,4,4,2,1,1) \bmod 2$



$$(q-t)F(7,7,7,6,5,5,4,4,4,4,2,1,1)$$

Stable triangle

Let $m = a_2 + 2a_3 + \cdots + (n-1)a_n = \deg(F(a_2, \ldots, a_n))$. Seems like there's stable triangle with vertices $(m, 0)$, $(0, m)$, $(\frac{m}{3}, \frac{m}{3})$.

Question

Does the stable triangle exist?

Stable triangle

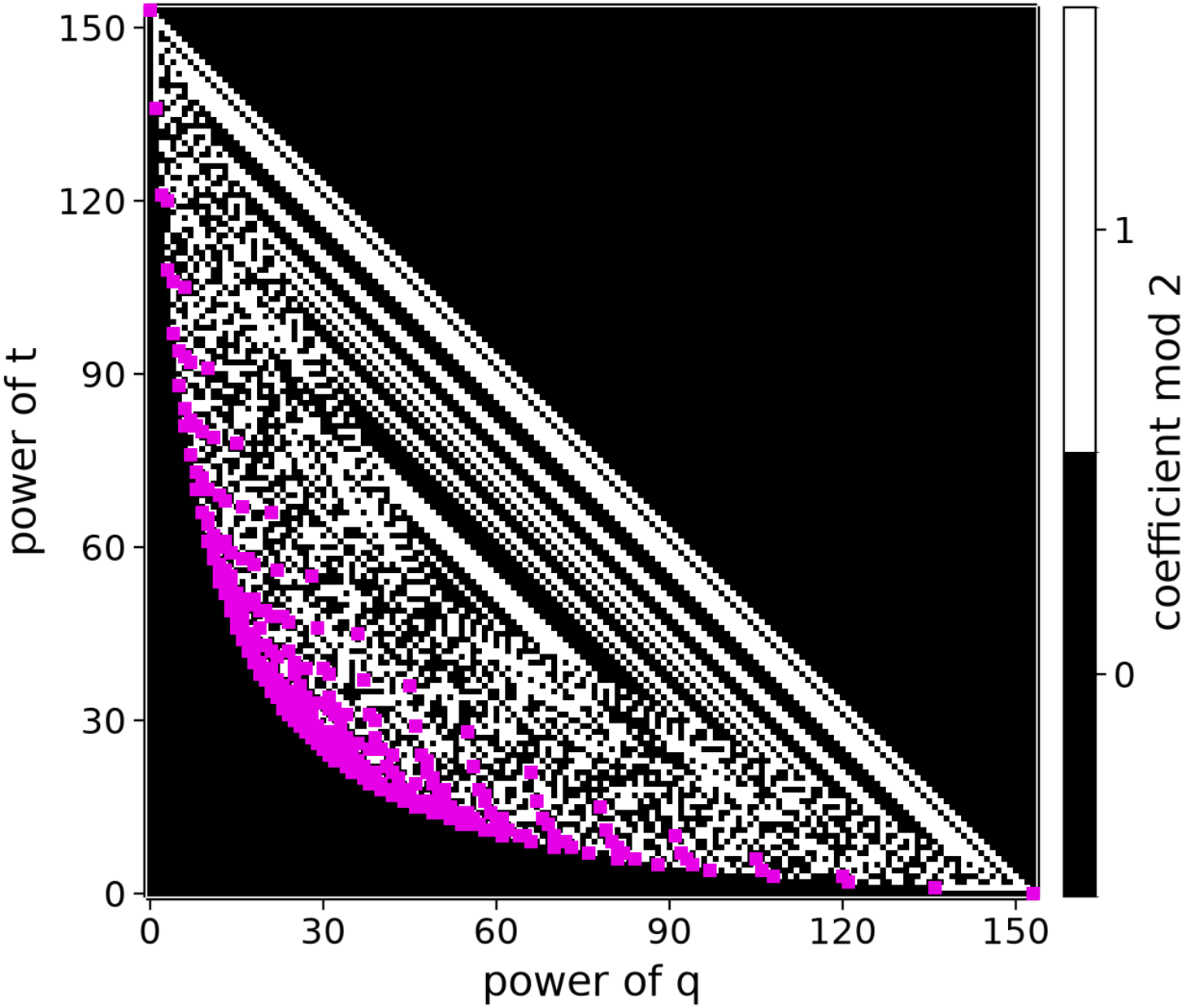
Let $m = a_2 + 2a_3 + \cdots + (n-1)a_n = \deg(F(a_2, \ldots, a_n))$. Seems like there's stable triangle with vertices $(m, 0)$, $(0, m)$, $(\frac{m}{3}, \frac{m}{3})$.

Question

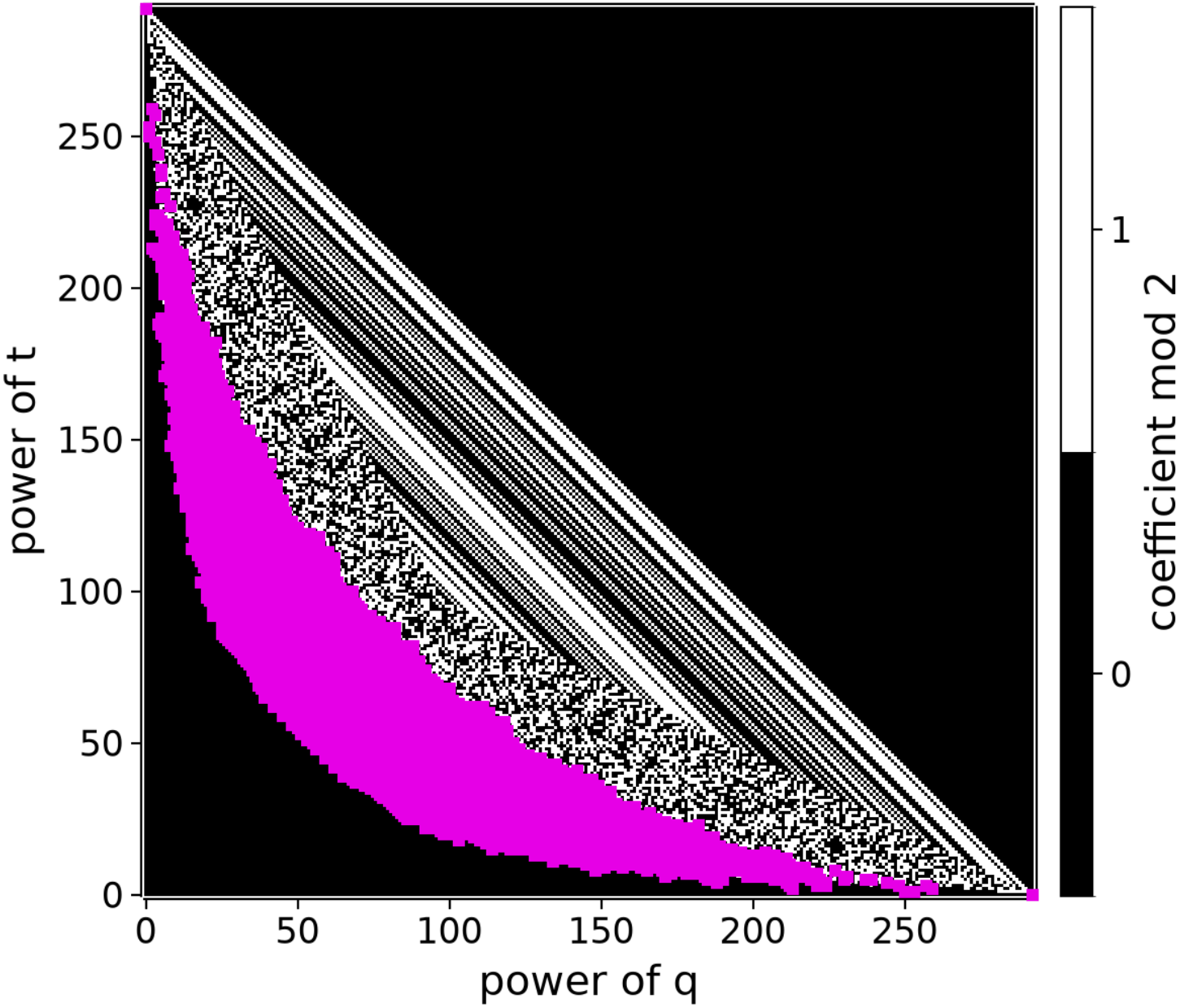
Does the stable triangle exist?

Theorem (Hawkes 2026)

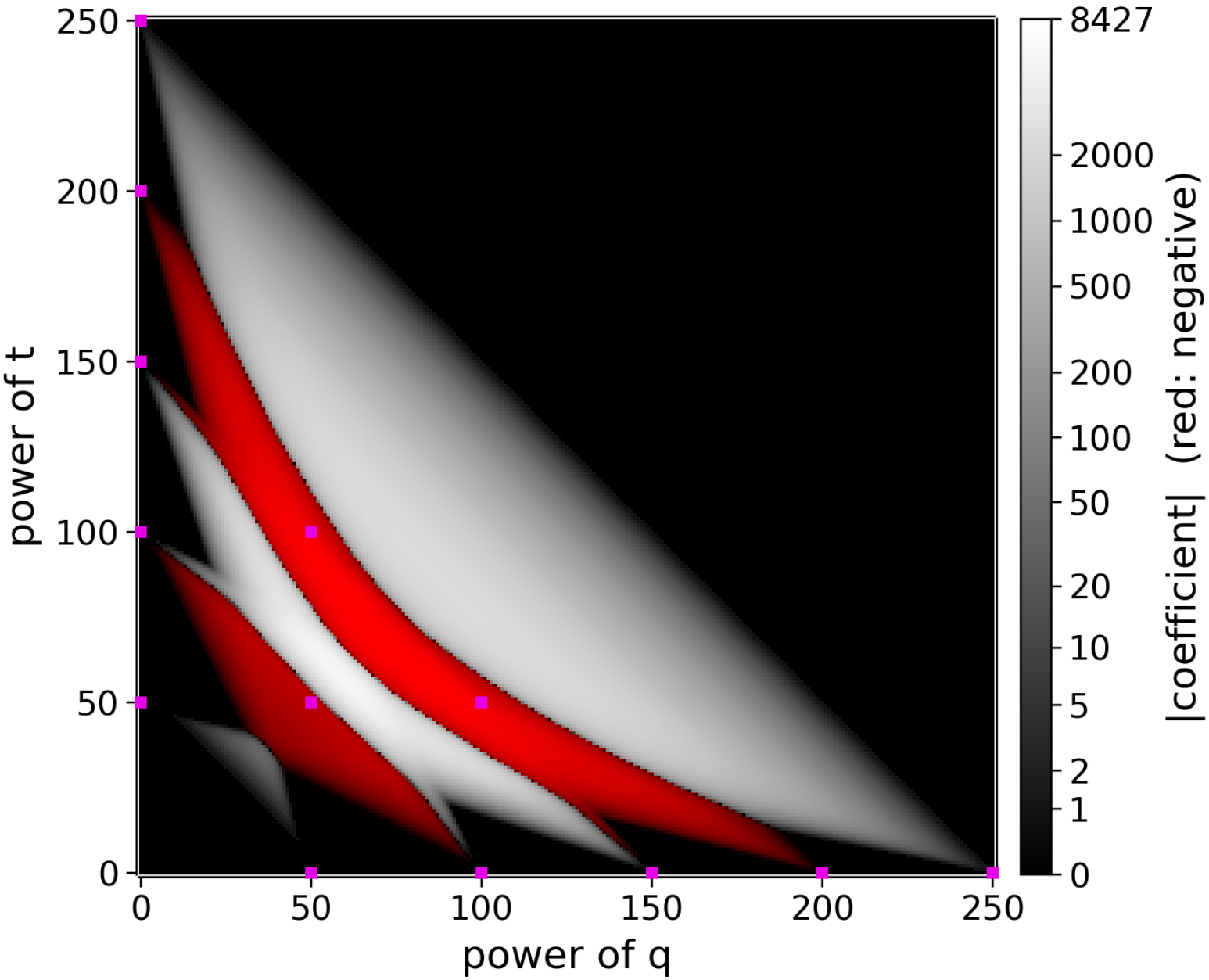
Yeah for the classical q, t-Catalan.



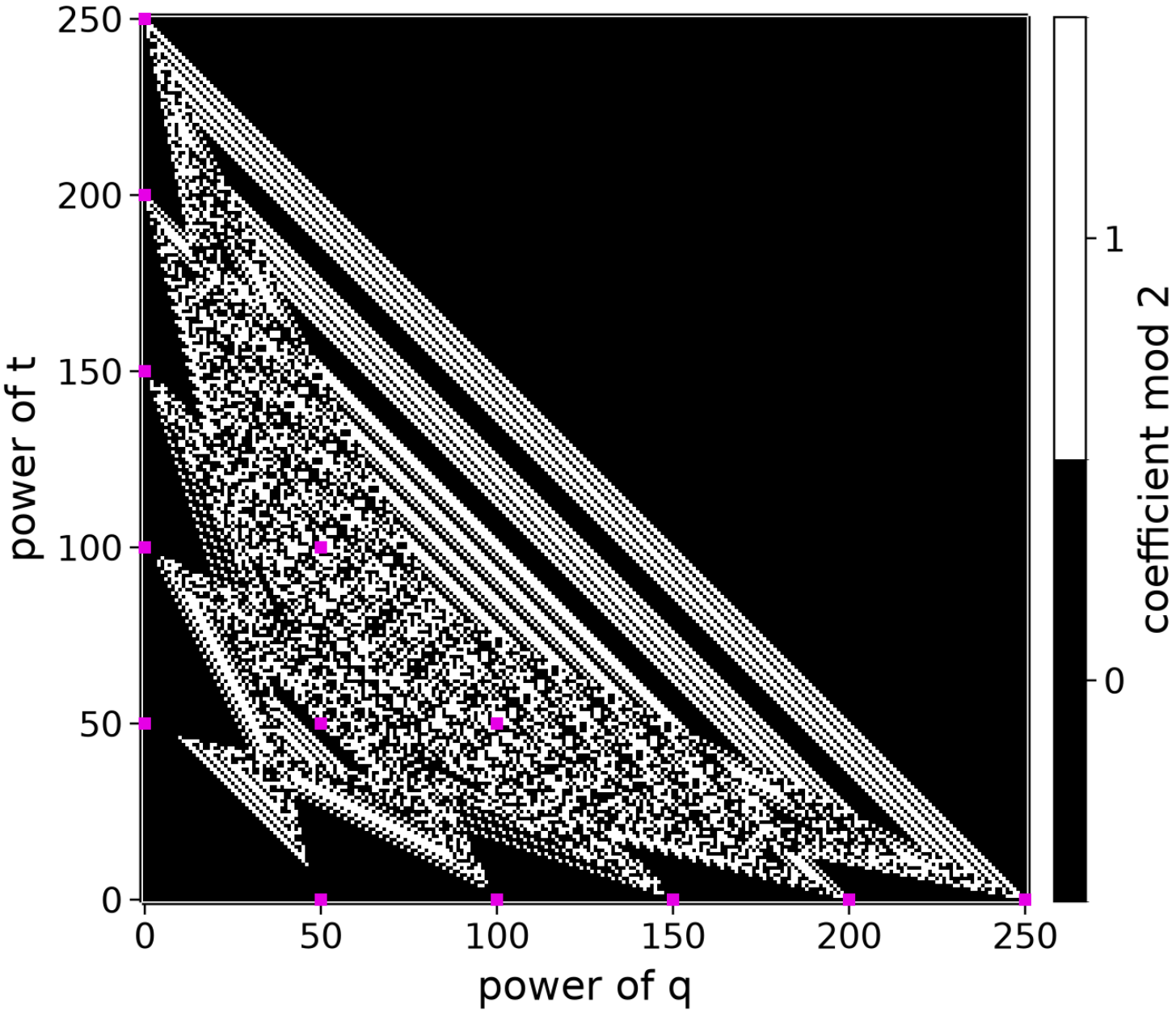
$F(1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1)$ mod 2



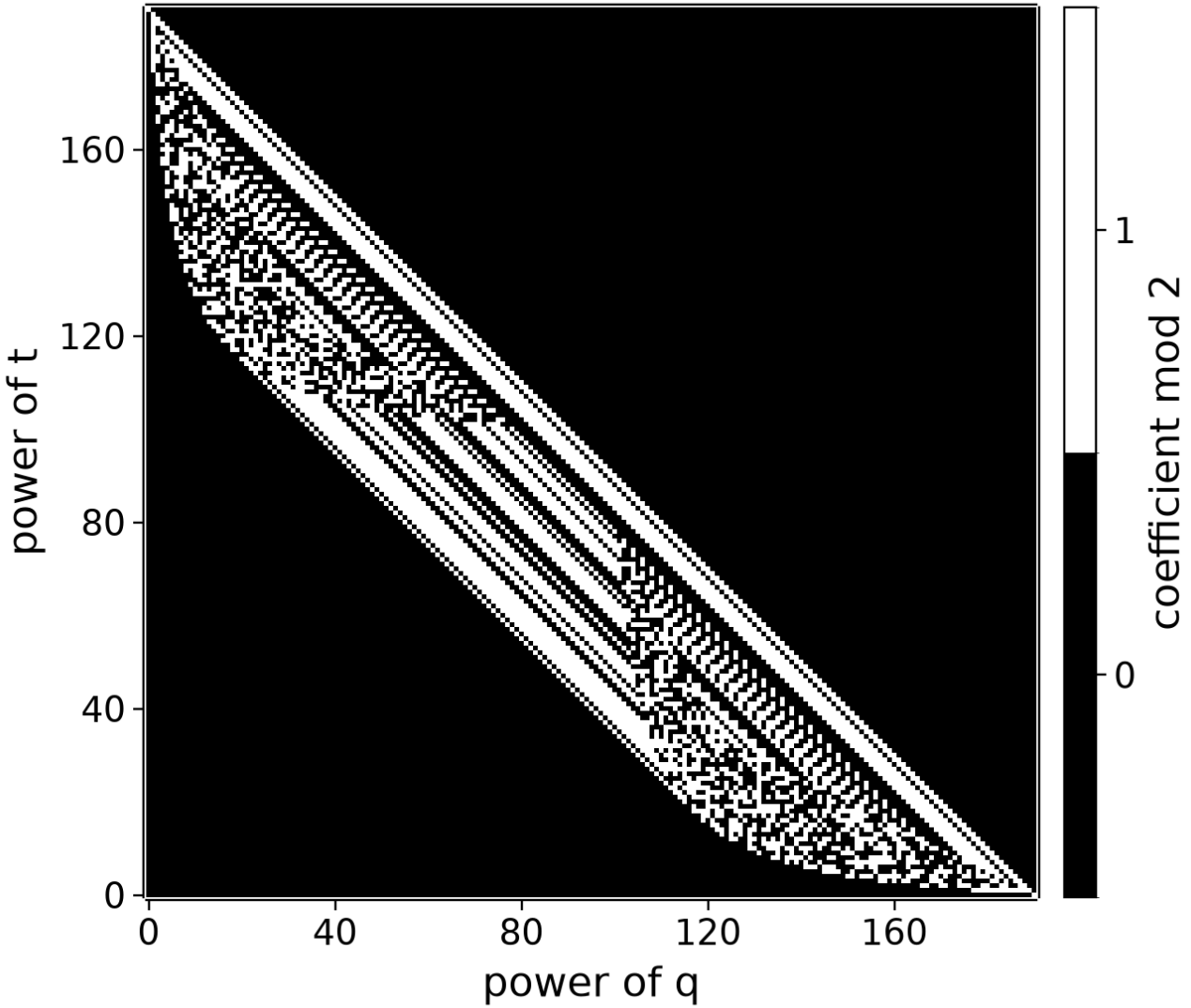
$F(3,1,4,1,5,9,2,6,5,3,5) \bmod 2$


250
200
150
100
50
0
power of t
0
50
100
150
200
250
power of q
8427
2000
1000
500
200
100
50
20
10
5
2
1
0
|coefficient| (red: negative)

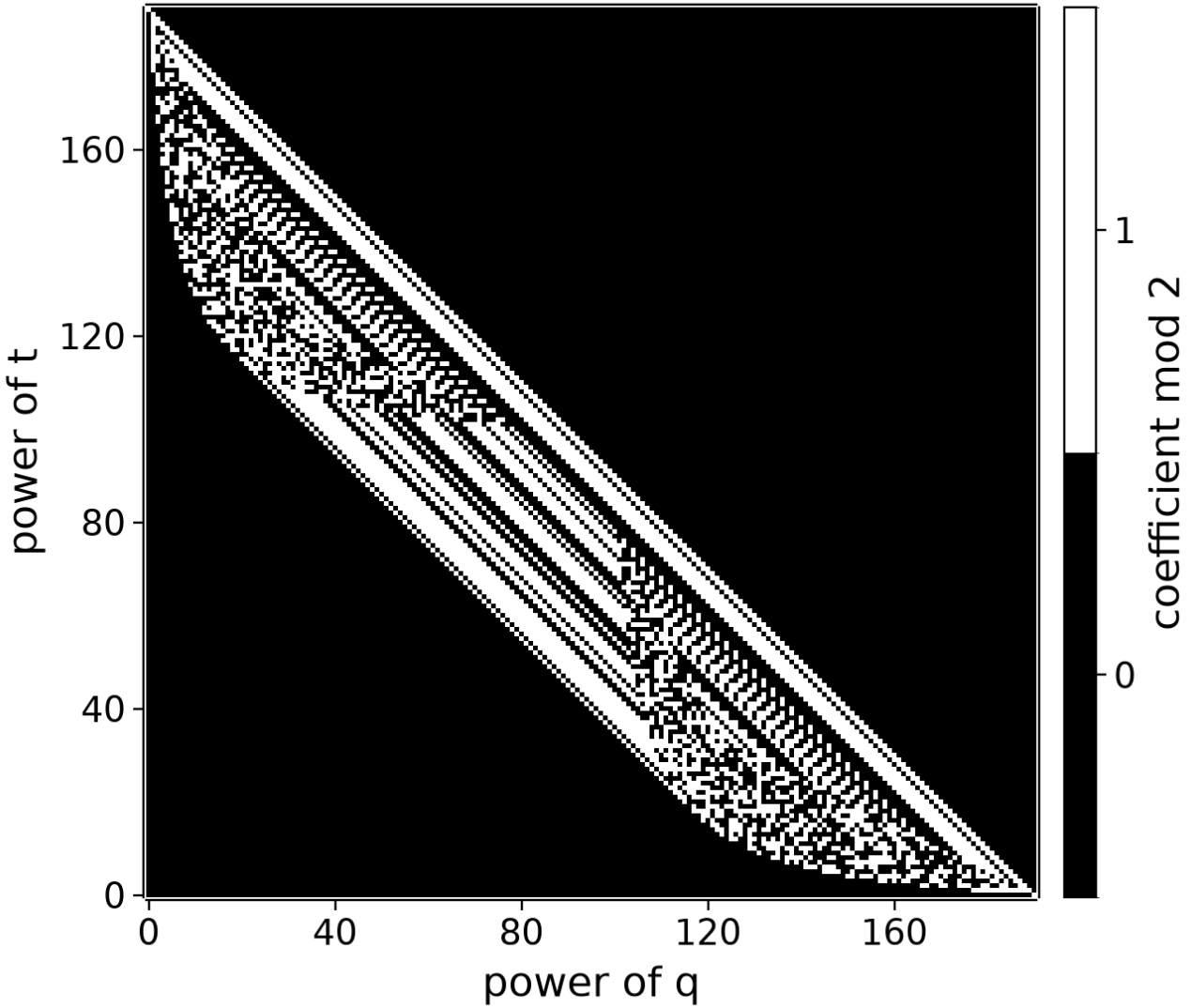
$F(0,0,0,0,50)$



$F(0,0,0,0,50) \bmod 2$



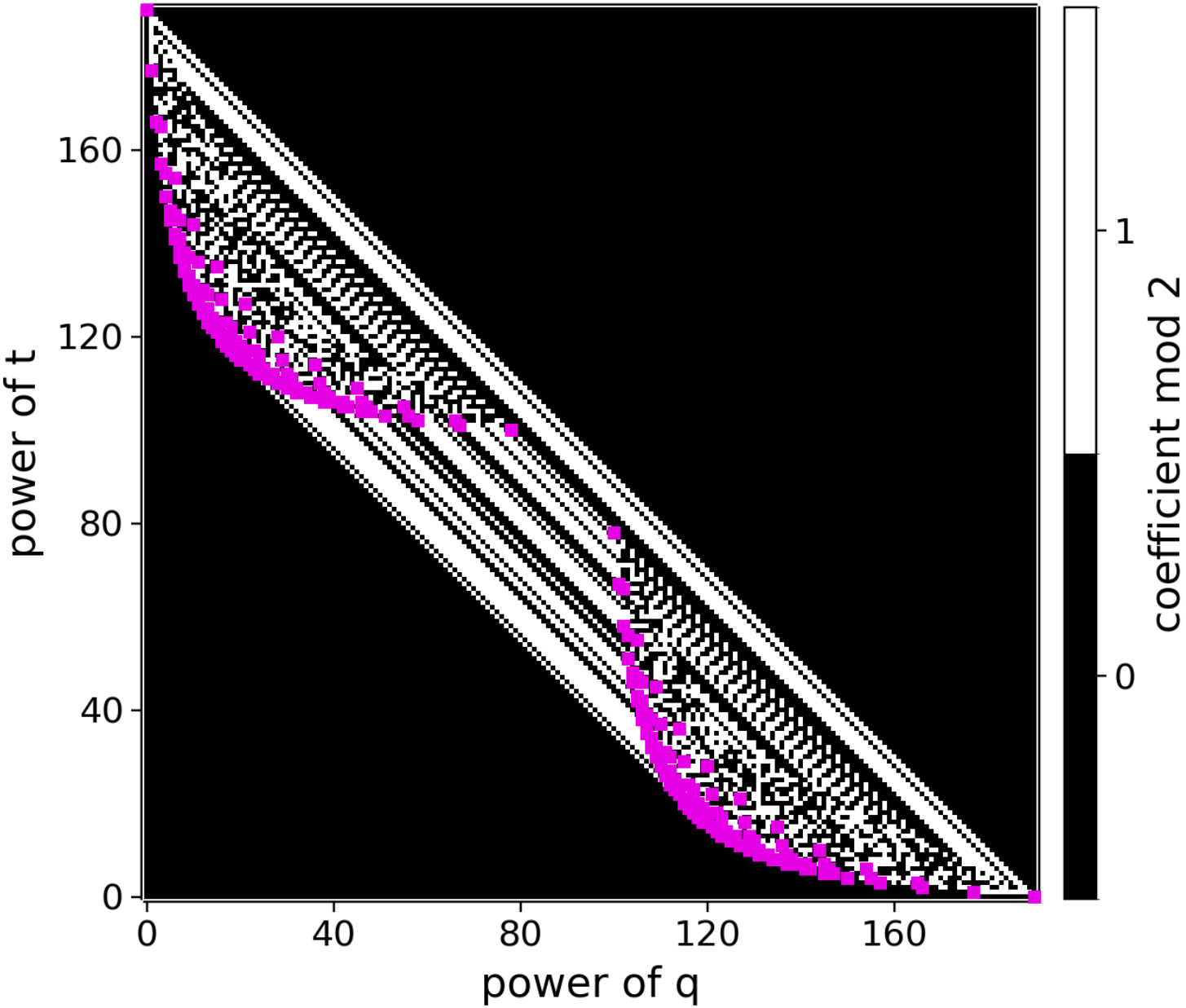
$F(100, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1) \bmod 2$



$F(100, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1) \bmod 2$

Question

What

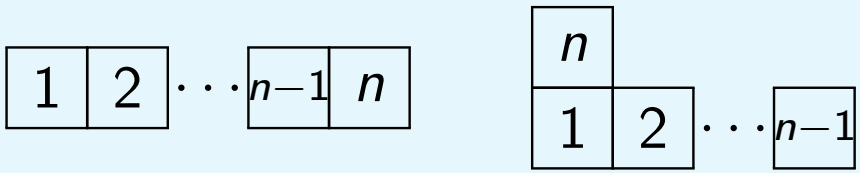


$F(100, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1)$ mod 2

Stable trapezoid

Theorem (L. 2026)

Suppose $a_3, \ldots, a_n \geq 1$. Mark every tableau point except for the four corresponding to

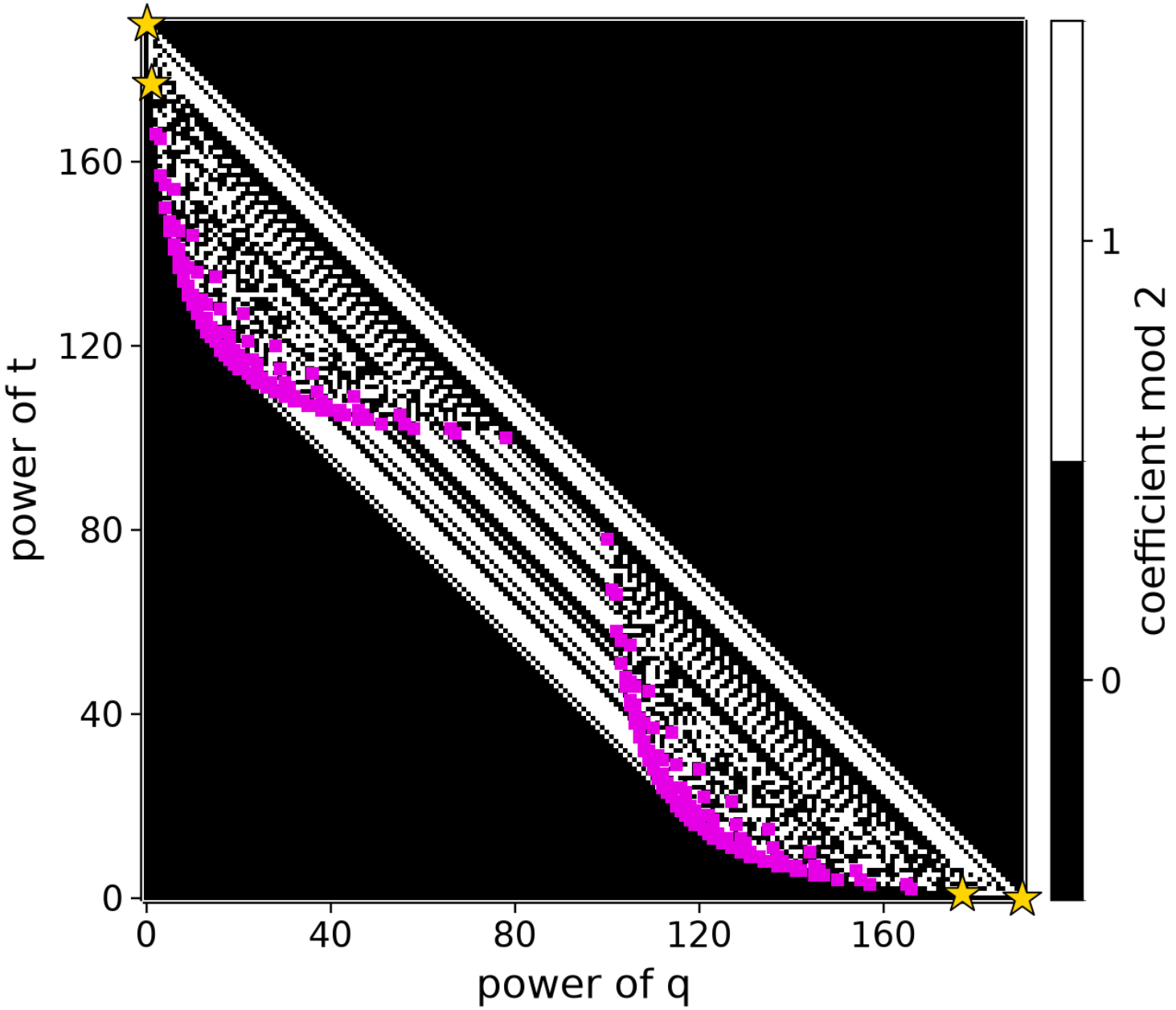


and their transposes. Diagonals above every marked point are stable. In this region, the $d-1$th highest diagonal has stable value

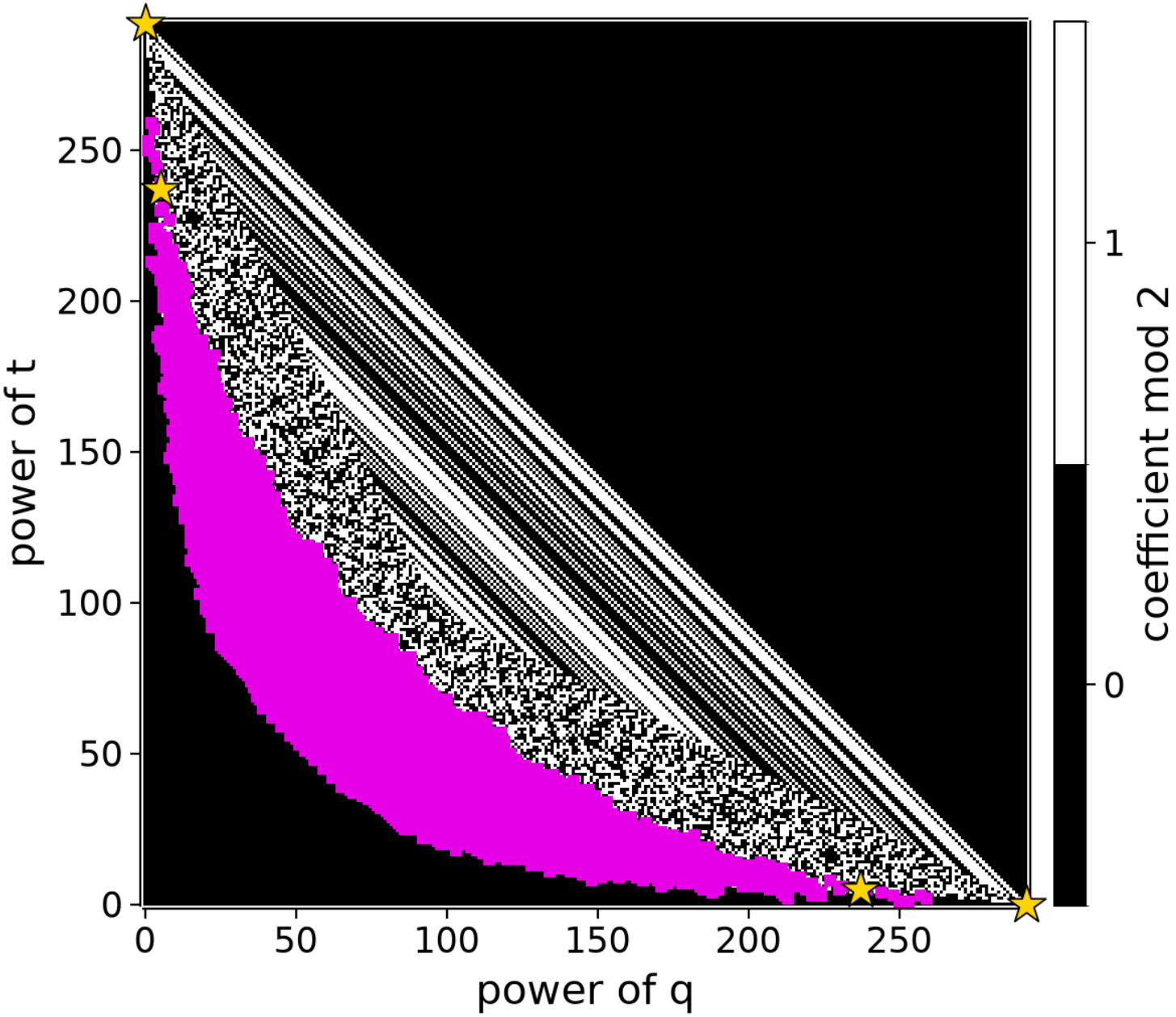
$$p(d; n-2) - p(d-(a_n+1)(n-2); n-2),$$

where $p(m;k)$ is # partitions of m with parts at most k.

This region is called the stable trapezoid.



$F(100, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1)$ mod 2



$F(3,1,4,1,5,9,2,6,5,3,5) \bmod 2$

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