A Bound on the Fractal Uncertainty Exponent for a Square Alphabet

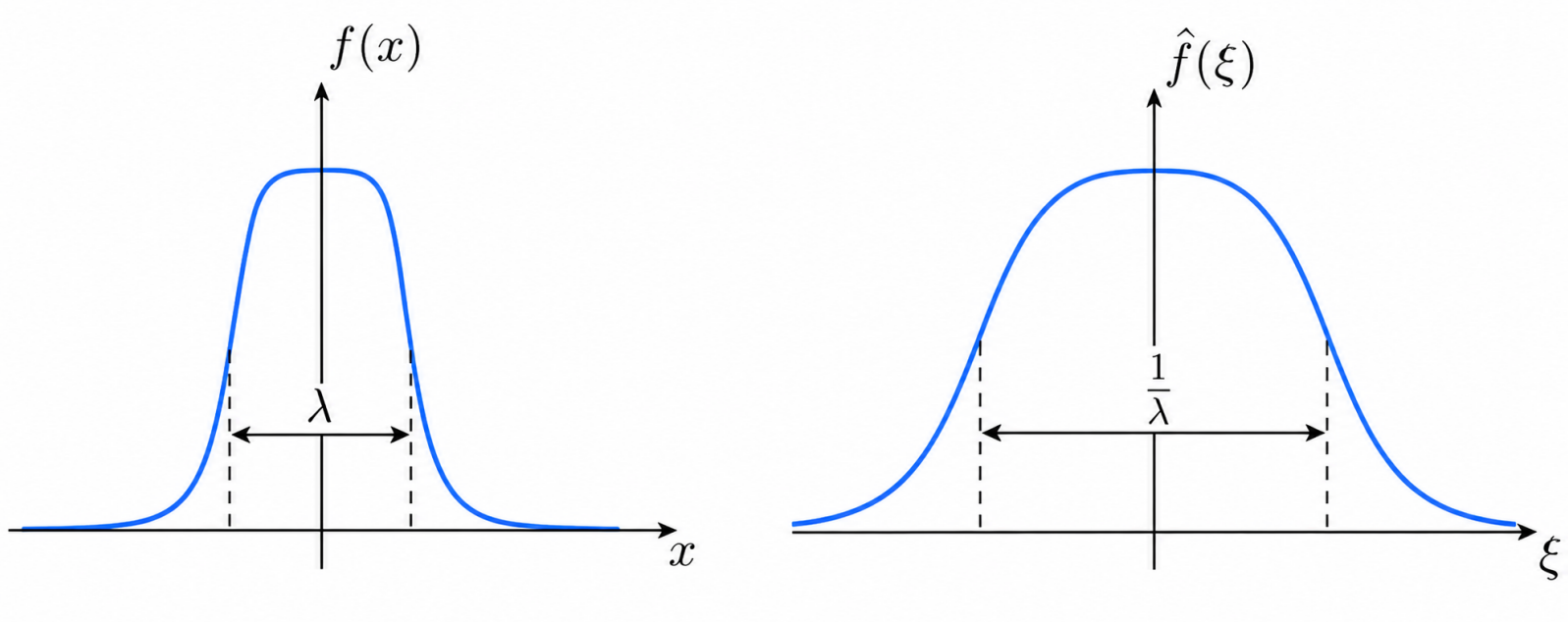
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Mentor: Alain Kangabire

MIT PRIMES Conference

October 2026

Motivation

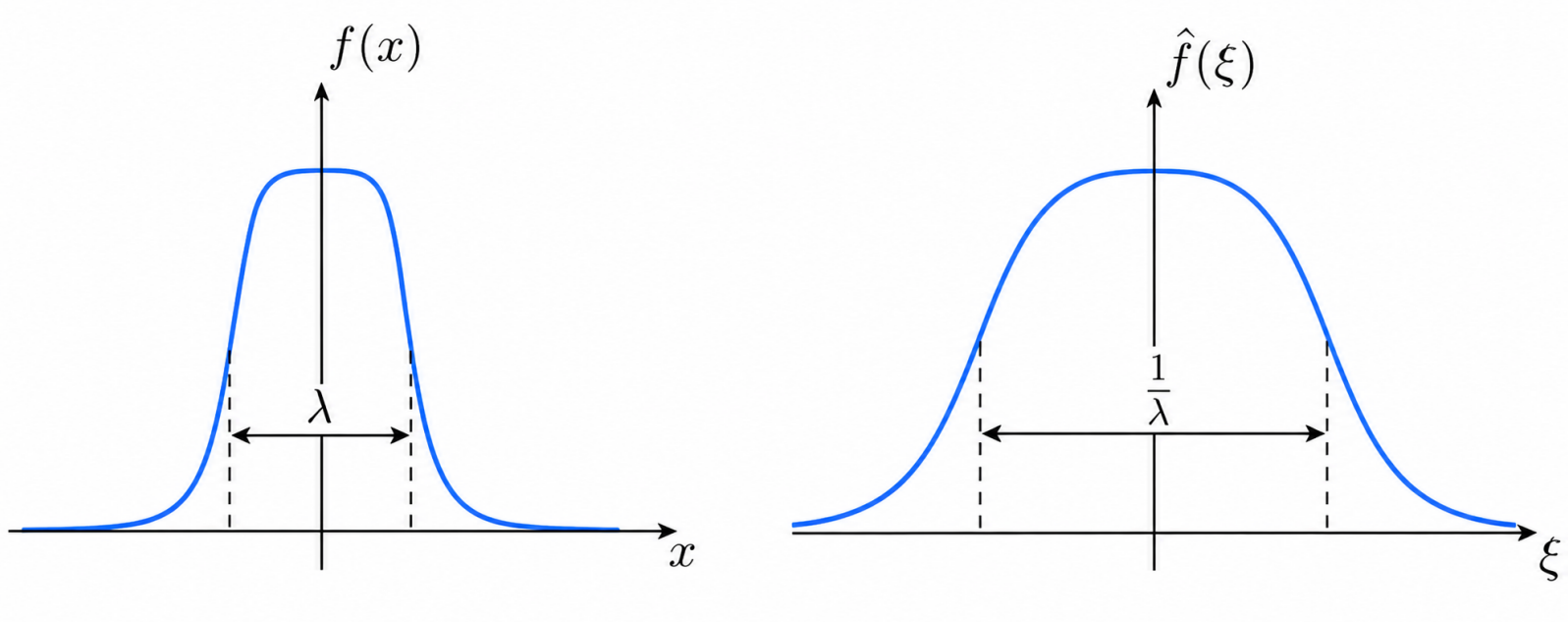
The **Uncertainty Principle** states that a wave cannot be simultaneously concentrated in both position and frequency.



Uncertainty Principle

Motivation

The **Uncertainty Principle** states that a wave cannot be simultaneously concentrated in both position and frequency.



Uncertainty Principle

The **fractal uncertainty principle** (FUP) says that a function and its **Fourier transform** cannot both be strongly concentrated on **fractal sets**.

Discrete Cantor Sets

We use **discrete Cantor sets** as our fractals, constructed from alphabets of allowed digits.

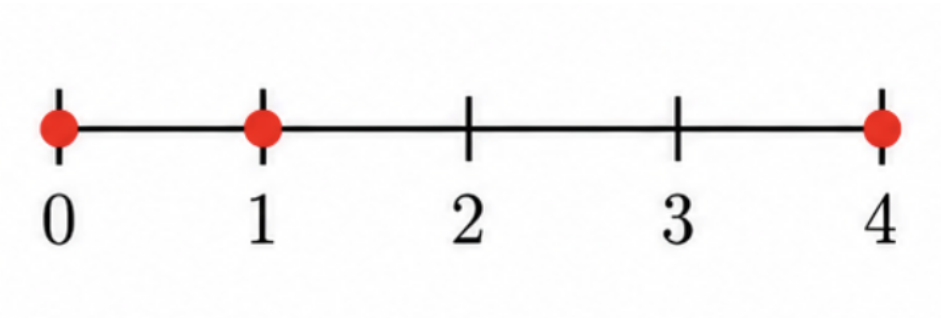
Discrete Cantor Sets

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An **alphabet** $\mathcal{A} \subset \{0, 1, \ldots, M-1\}$ is a set of allowed base-M digits. In this project, we study the square alphabet.

Example

Let $M = 5$ and $\mathcal{A} = \{0^2, 1^2, 2^2\}$.



Discrete Cantor Sets

Choose base M and alphabet $\mathcal{A} \subset \{0, 1, \ldots, M-1\}$ of allowed digits.

Definition

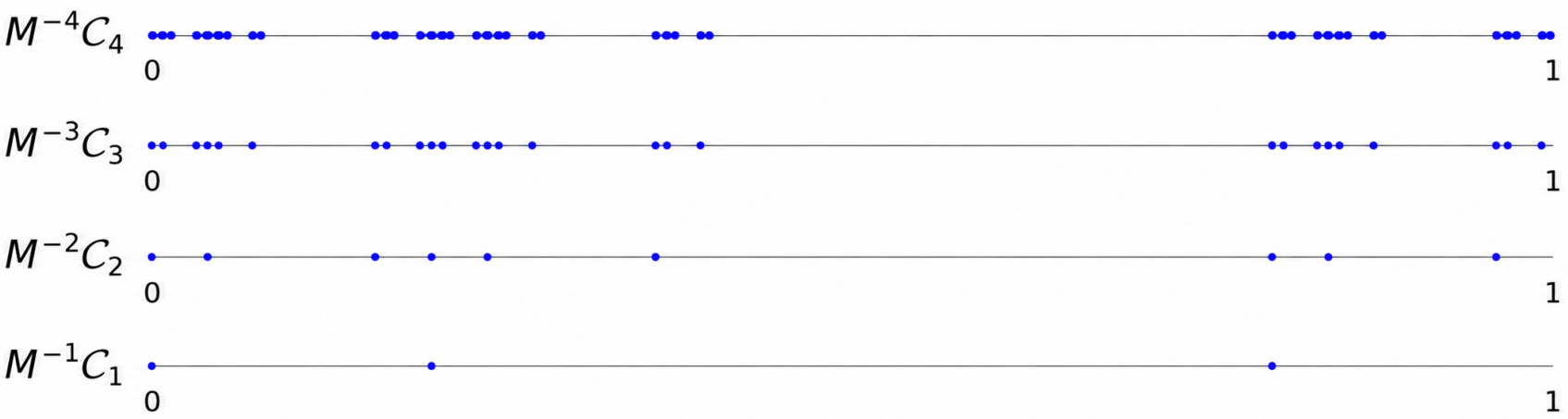
$$\mathcal{C}_k = \{a_0 + a_1 M + \cdots + a_{k-1} M^{k-1} : a_j \in \mathcal{A}\} \subset \mathbb{Z}_{M^k}$$

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$$M = 5 \text{ and } \mathcal{A} = \{0, 1, 4\}$$

The **Hausdorff dimension** of these Cantor sets is given by $\delta = \frac{\log |\mathcal{A}|}{\log M}$.

Discrete Fourier Transform

Suppose that we have a function

$$f : \{0, \cdots, N-1\} \to \mathbb{C}.$$

We view f as a vector in $\mathbb{C}^N$. The discrete Fourier transform on $\mathbb{C}^N$, where $N \in \mathbb{N}$, is defined as

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$$\hat{f}(j) = \mathcal{F}_N(f)(j) = \frac{1}{\sqrt{N}} \sum_{l=0}^{N-1} f(l) e^{-\frac{2\pi ijl}{N}}.$$

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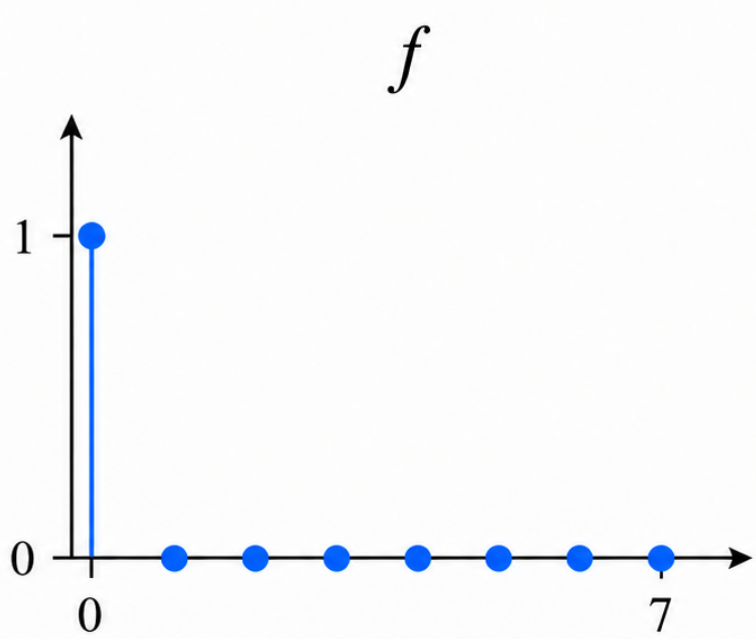
The norm $\|f\|$ for functions $f \in \mathbb{C}^N$ is defined as

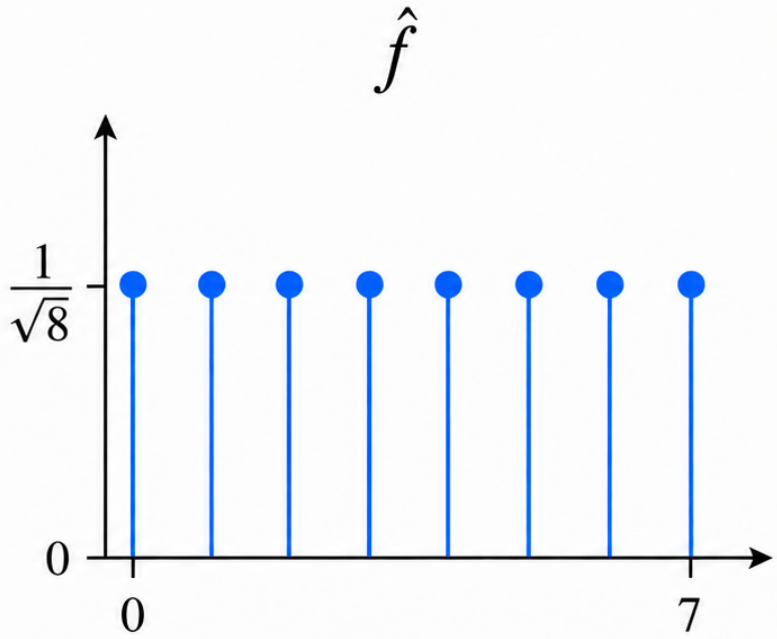
Definition

$$\|f\|^2 = \sum_{l=0}^{N-1} |f(l)|^2.$$

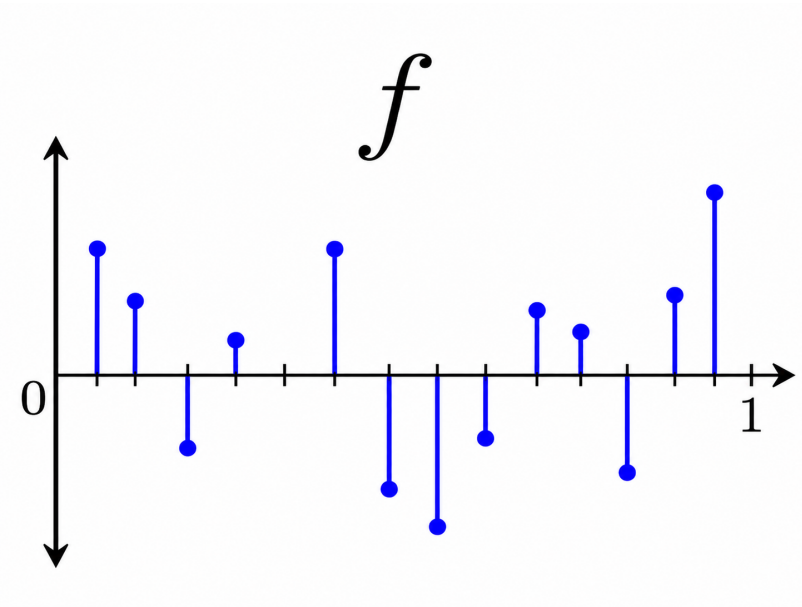
Example of Fourier transform

Let $f = (1, 0, 0, 0, 0, 0, 0, 0)$.

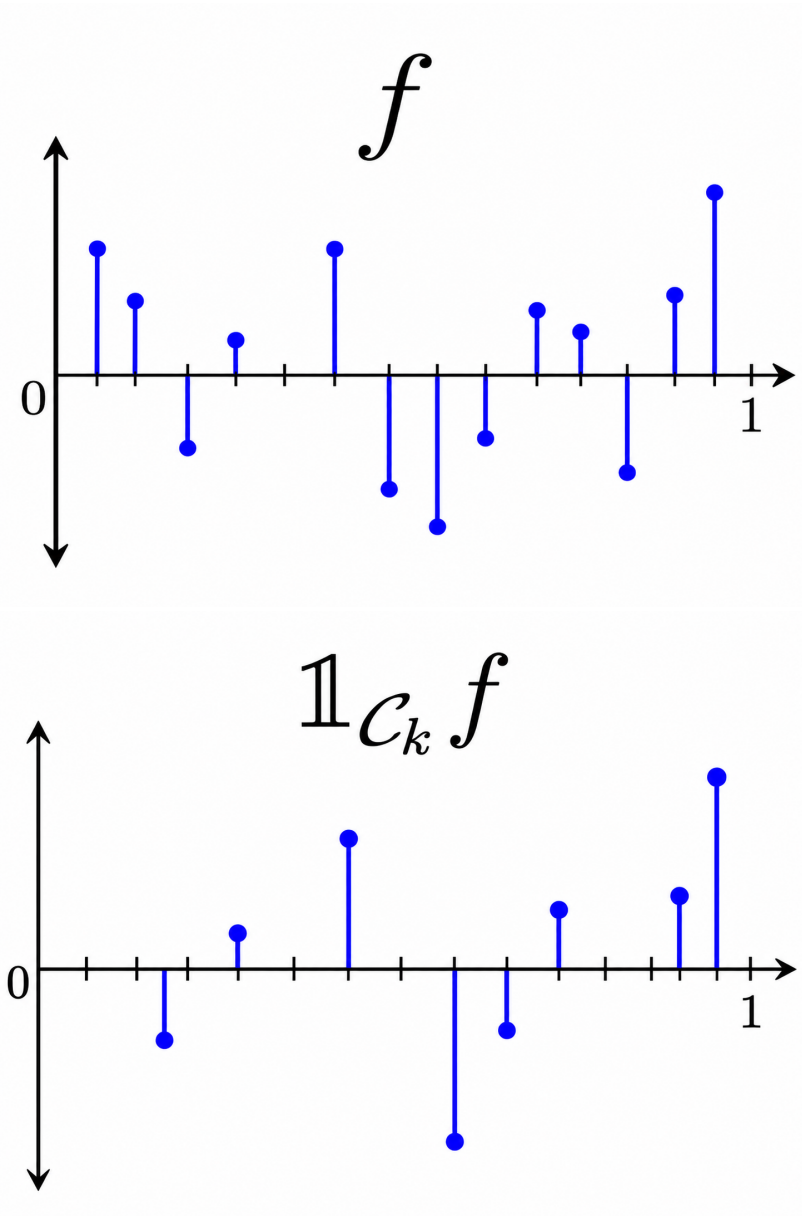




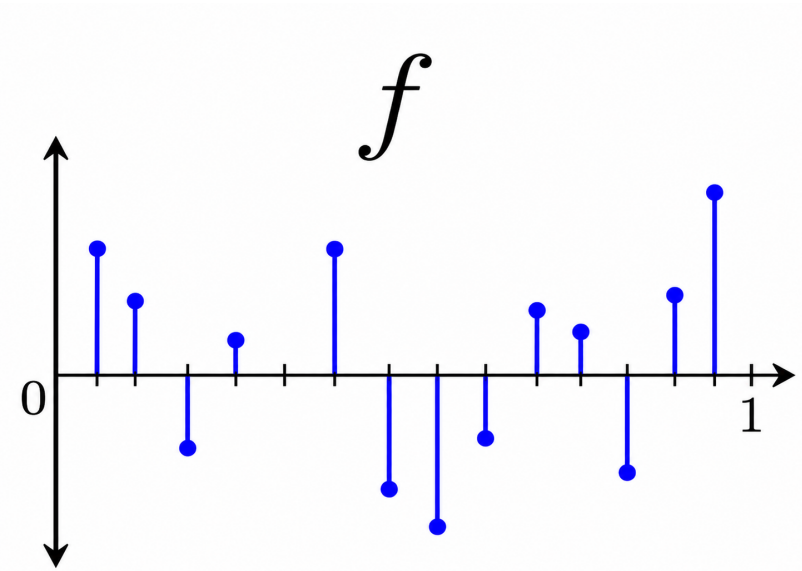
Discrete Cantor sets and Fourier transform

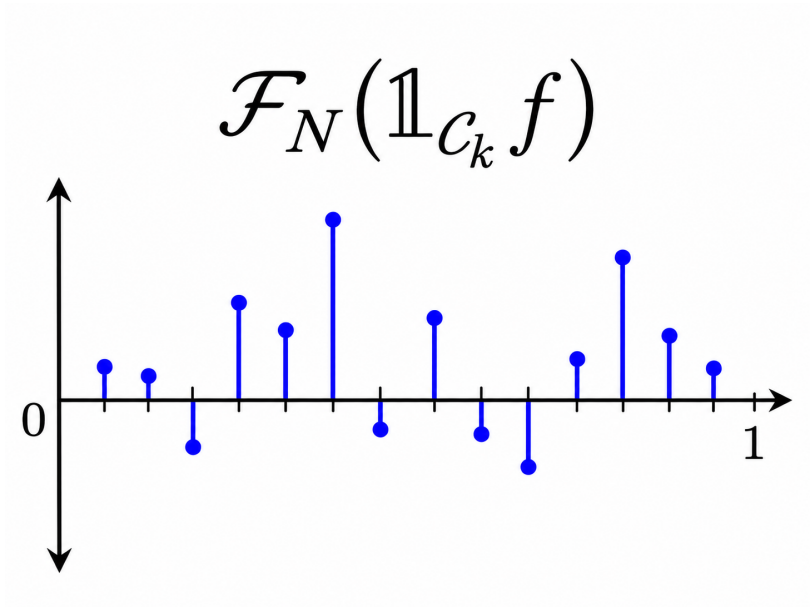


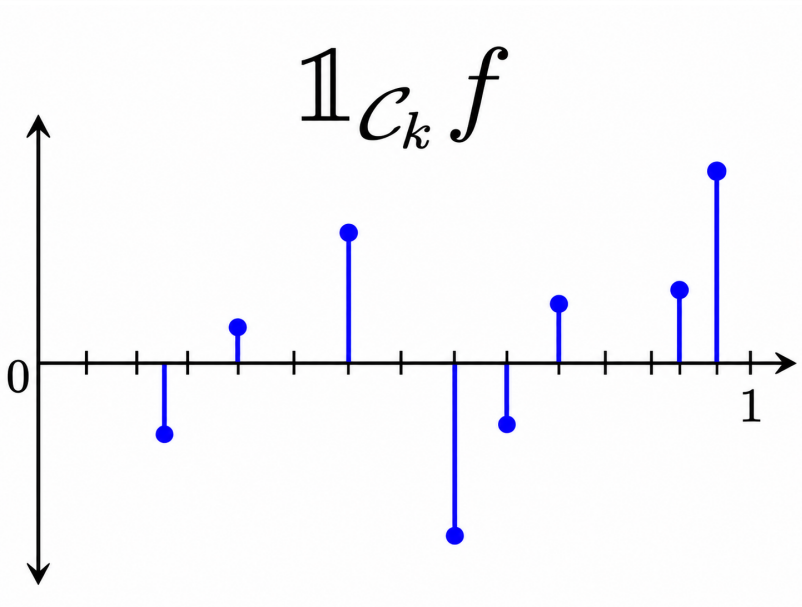
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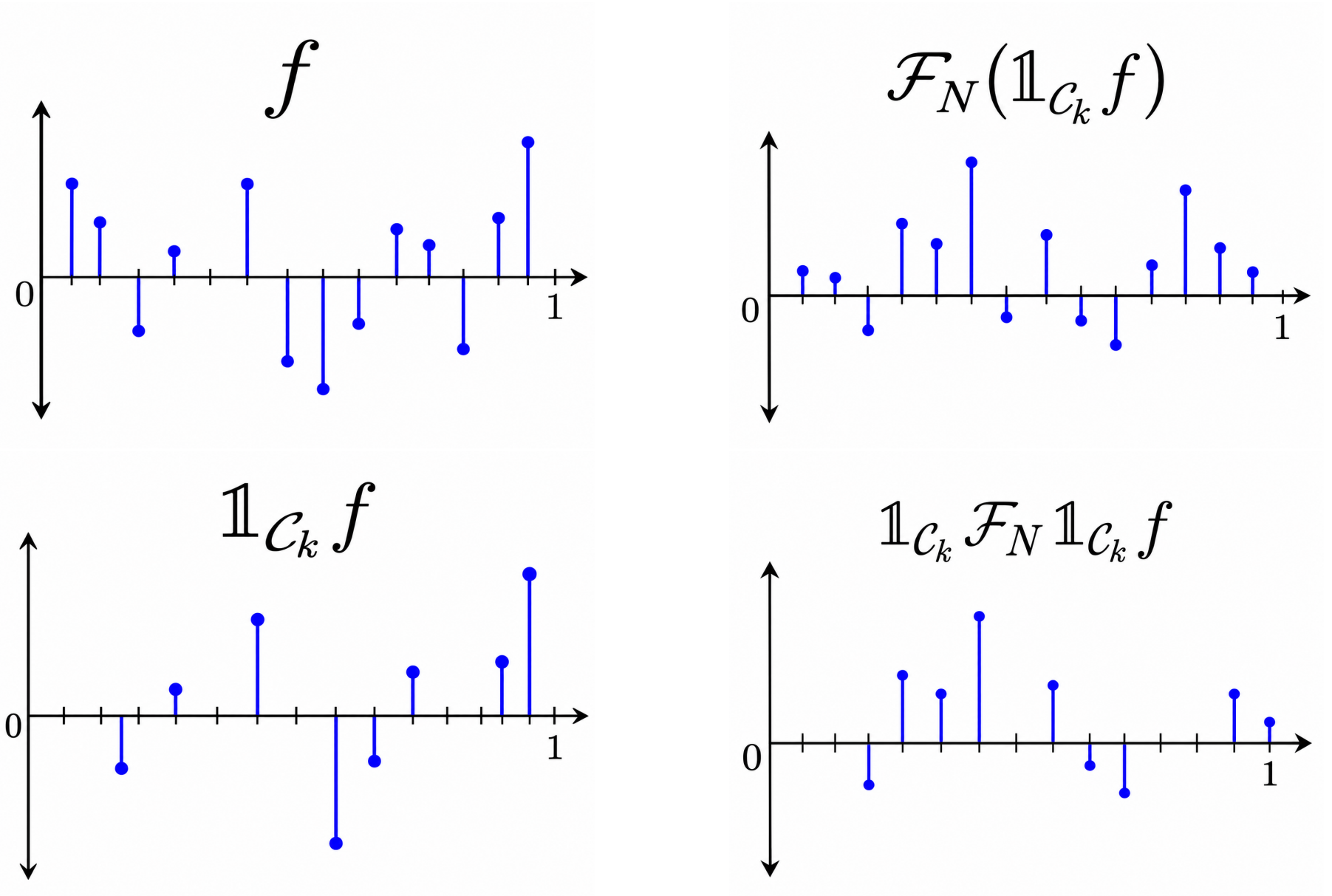
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Discrete Cantor sets and Fourier transform



Fractal Uncertainty Principle (FUP)

The fractal uncertainty principle measures how difficult it is for a function and its Fourier transform to both remain concentrated on the same Cantor set.

Theorem (FUP Dyatlov–Jin)

There exists $\beta > \max(0, \frac{1}{2} - \delta)$ *such that for any* $\epsilon > 0$

$$\|\mathbb{1}_{\mathcal{C}_k} \mathcal{F}_N \mathbb{1}_{\mathcal{C}_k}\| \leq C_\epsilon N^{-\beta+\epsilon}, \quad N = M^k.$$

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Main Goal: A larger β value signifies faster decay and stronger uncertainty.

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Does FUP still hold with **dilated** *Cantor sets?*

Dilated Cantor Sets

Instead of studying original Cantor sets $\mathcal{C}_k$, we stretch these sets by a factor of $\alpha \in [1, M)$.

Definition

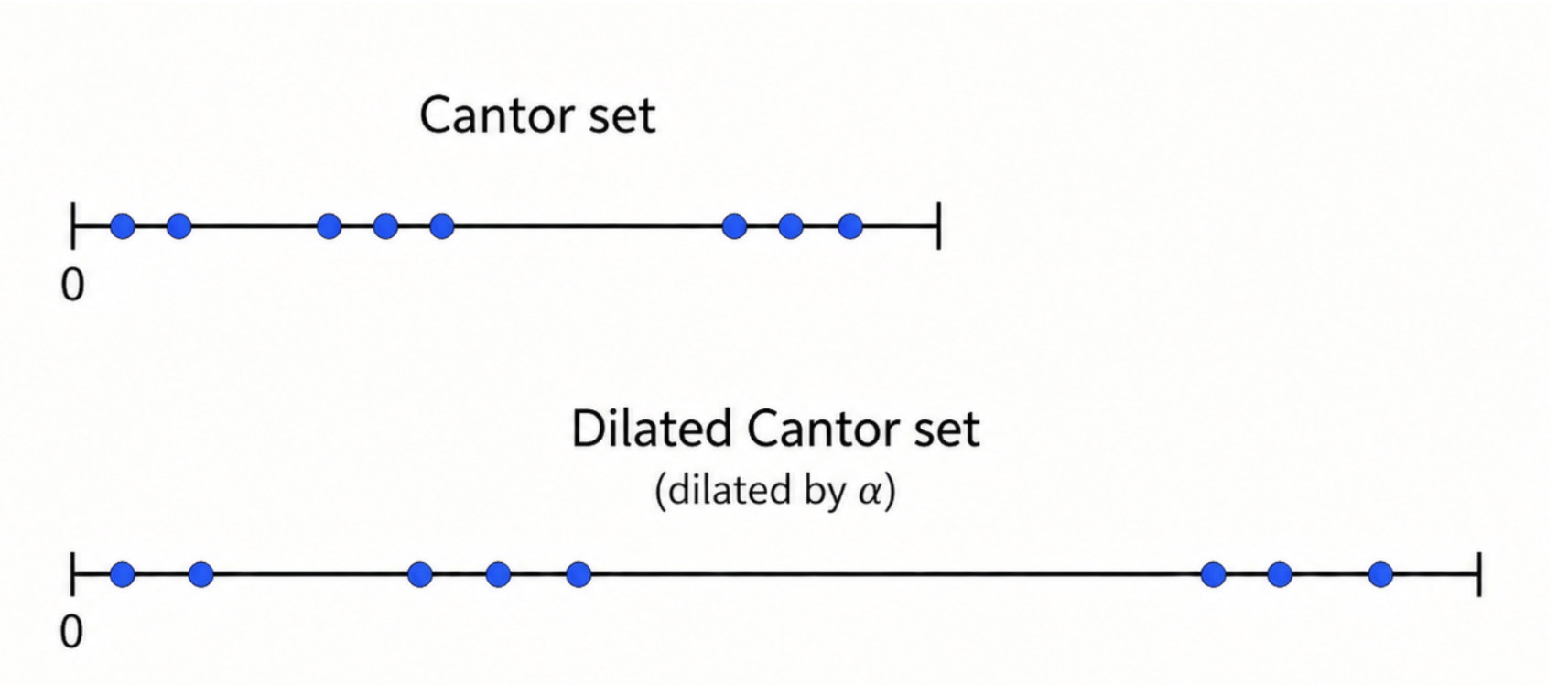
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Results

Theorem (Li)

Let $\epsilon > 0$, $\frac{1}{4} \geq \delta > 0$, *and* $\mathcal{A} = \{0^2, 1^2, \ldots, \left(M^\delta - 1\right)^2\}$. *Then,*

$$\left\| \mathbb{1}_{\mathcal{C}_k(N)} \mathcal{F}_N \mathbb{1}_{\mathcal{C}_k(N)} \right\| \leq N^{-\left(\frac{1}{2} - X_1 \delta\right) + \epsilon}, \quad N = \alpha M^k$$

for all but a small proportion of values of α *that vanishes as* $M \to \infty$, *and* $0 < X_1 < 1$.

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This theorem improves on the previous lower bound

$$\beta > \frac{1}{2} - \delta,$$

which has total possible improvement of $\frac{\delta}{2}$. This theorem improves this previous lower bound by $(1 - X_1)\delta$.

Results

Theorem (Li)

Let $\epsilon > 0$, $\frac{1}{4} \geq \delta > 0$, *and* $\mathcal{A} = \{0^2, 1^2, \ldots, \left(M^\delta - 1\right)^2\}$. *If* $\frac{\alpha}{M} = \frac{a}{b} \in \mathbb{Q}$ *and* $c\left(M^\delta - 1\right) \leq b \leq C\left(M^\delta - 1\right)$ *for some constants* $c, C > 0$, *then*

$$\left\| \mathbb{1}_{\mathcal{C}_k(N)} \mathcal{F}_N \mathbb{1}_{\mathcal{C}_k(N)} \right\| \leq N^{-\left(\frac{1}{2} - X_2\delta\right)+\epsilon}, \quad N = \alpha M^k,$$

where $0 < X_2 < 1$.

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where $0 < X_2 < 1$.

This theorem gives the stronger lower bound

$$\beta \geq \frac{1}{2} - X_2\delta - \epsilon.$$

This improves on the general lower bound of $\frac{1}{2} - \delta$ on β by $(1 - X_2\delta)$.

Details of Proof

Theorem (Kangabire)

If $N = \alpha M^k \in \mathbb{N}$ *for* $M, k \in \mathbb{N}$ *and* $\alpha \in \mathbb{Q} \cap [1, M)$, *then there exists a global constant* C *such that*

$$\|\mathbb{1}_{\mathcal{C}_k(N)} \mathcal{F}_N \mathbb{1}_{\mathcal{C}_k(N)}\|^2 \leq \alpha C G^k$$

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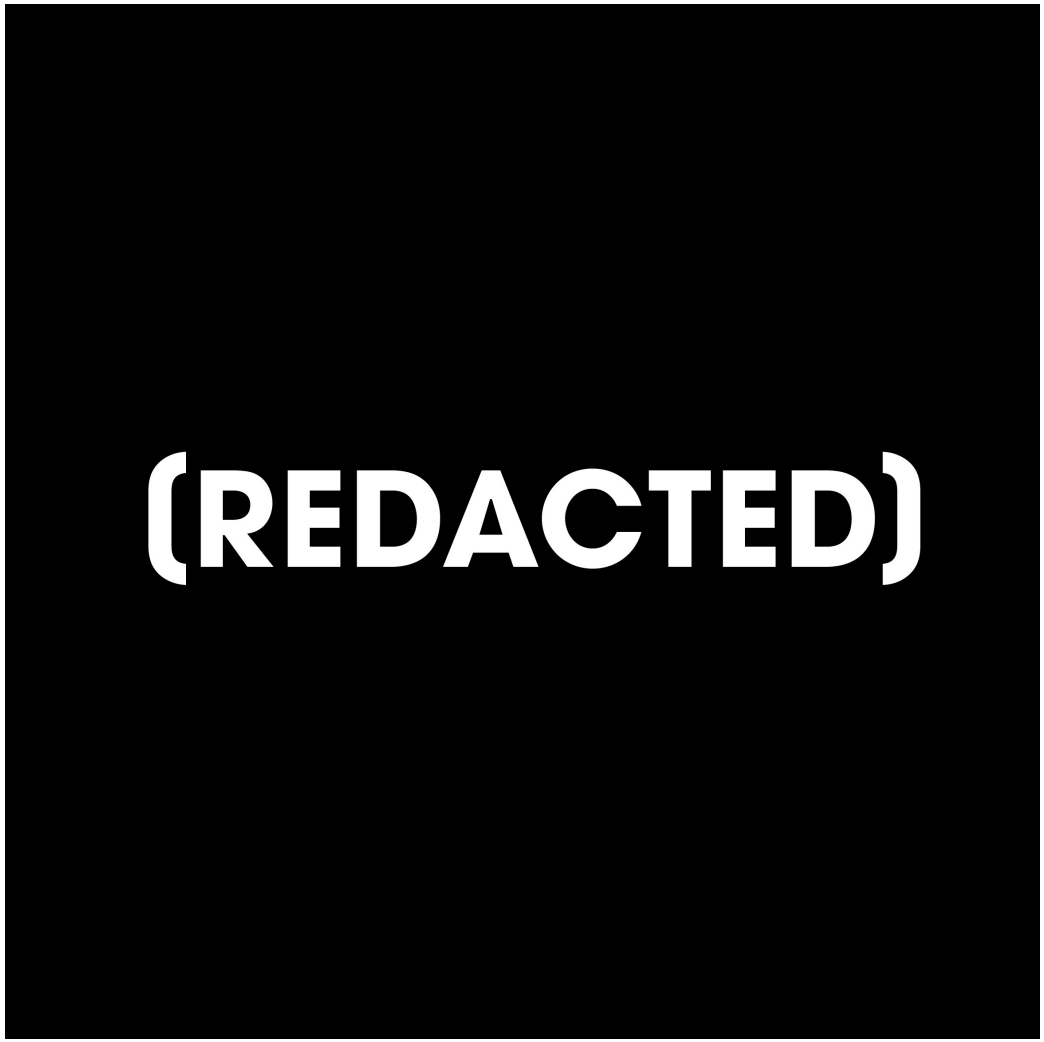
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Definition

$$G = M^{-1} \sup_{x \in \mathbb{R}} \sum_{q=0}^{L} \left(\sup_{y \in J_{x,q}} \left| \sum_{l=0}^{L} e^{-2\pi i l^2 y} \right| \right), \quad L = M^{\delta} - 1.$$

$$J_{x,q} = x + \gamma q^2 + [0, p], \quad \gamma = \frac{\alpha}{M}, \quad p > 0 \quad \text{(small)}$$

Details of Proof



Until further notice

Future work

- Find a optimal lower bound for β
- Consider alphabets $\mathcal{A}_d = \{0^d, 1^d, ..., (M^\delta - 1)^d\}, \quad d \geq 3$
- Consider any alphabet $\mathcal{A}$

Acknowledgements

I would like to thank my PRIMES mentor, Alain Kangabire, for his valuable advice and guidance. I would also like to express my gratitude to the MIT PRIMES-USA program for giving me the opportunity to conduct this research.

References

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- Alain Kangabire, *Bounds on the Fractal Uncertainty Exponent and a Spectral Gap*, arXiv:2406.06815v1 [math.AP], 2024.