T-equivariant modules over the ring $k[x_1, x_2, \ldots, x_n][y_1, y_2, \ldots, y_{n-1}]/(x_i y_i)_{i=1}^{n-1}$

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Overview

1. Introducing some machinery
2. Motivating the $n = 2$ case
3. Setup of the problem

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R-modules

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Definition

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Example

Over a field k, a k-module is just a vector space V. The $\mathrm{Hom}_k(V, V')$ are exactly linear transformations.

Exact sequences

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$$A \xrightarrow{f} B \xrightarrow{g} C.$$

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Definition

This sequence is *exact* at B if $\ker g = \operatorname{im} f$.

Or equivalently, if $\operatorname{im} f \subset \ker g$, then $B/f(A) \cong \operatorname{im}(g)$.

Exact sequences (cont.)

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Example

If V, V', W are vector spaces and

$$0 \longrightarrow V \xrightarrow{f} W \xrightarrow{g} V' \longrightarrow 0.$$

Then $V \oplus V' \cong W$ is the only short exact sequence (up to equivalence).

Vector spaces are nice because they have a basis.

Projective R-modules

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However ...

- many R-modules are not free.
- many short exact sequences don't split:

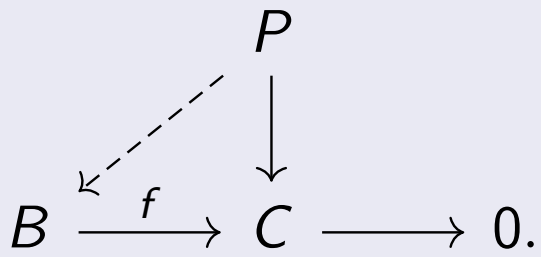
$$0 \longrightarrow \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow 0.$$

There is a specific type of R-module that is very well-behaved.

Projective R-modules

Definition

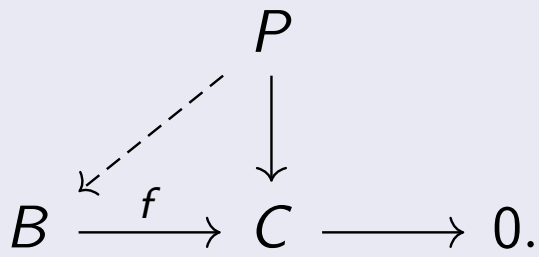
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Projective R-modules

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Projective modules have very similar behavior to vector spaces.

- P is projective if and only if there is P' with $P \oplus P'$ free.
- Every short exact sequence

$$0 \longrightarrow A \longrightarrow B \longrightarrow P \longrightarrow 0,$$

splits.

Projective resolutions

To understand general modules M, we approximate them via projective modules:

$$\cdots \longrightarrow P_2 \longrightarrow P_1 \longrightarrow P_0 \xrightarrow{\epsilon} M \longrightarrow 0,$$

such that it is exact at all terms. Fortunately, every R-module M can be resolved in this way!

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Used to calculate $\mathrm{Ext}^i(A, B)$ "higher extension groups" of A and B in the category of R-modules.

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Apply $\mathrm{Hom}(-, B)$:

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Then $\mathrm{Ext}^i(A, B) := \ker(d_i')/\operatorname{im}(d_{i-1}')$.

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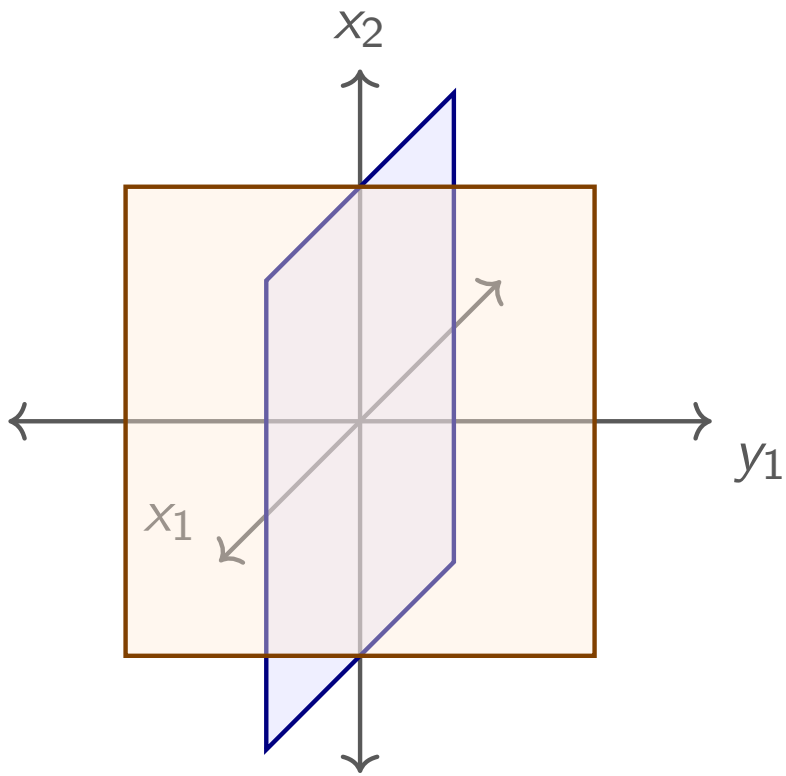
Fix the ring $R = k[x_1, x_2, y_1]/(x_1 y_1)$[1]. The *spectrum* $\mathrm{Spec}(R)$ of the ring has the structure of a topological space.

[1] k is algebraically closed.

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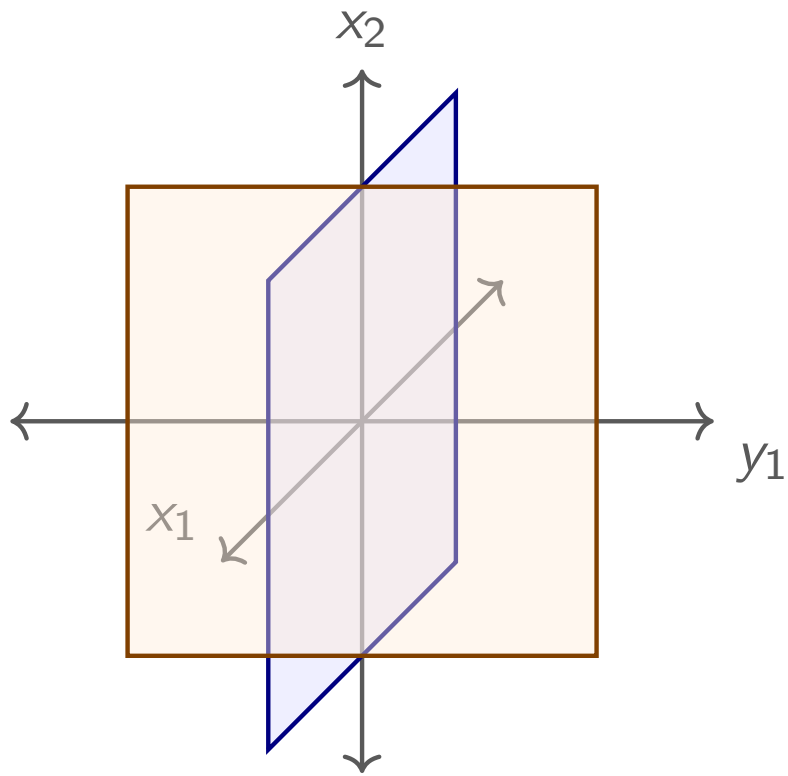


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- **Plane $(x_1 = 0)$:**
  $(x_1, y_1 - u, x_2 - v)$
- non-maximal prime ideals "glue" closed points together.



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The torus action

Consider the 2-dimensional torus $T = \mathbb{G}_m^2 = \mathrm{Spec}(k[t_1, t_2, t_1^{-1}, t_2^{-1}])$. Think of T as points $(a, b) \in (k^\times)^2$.

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A torus action in this situation can be thought of as a map:

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i.e. for an element $(a, b) \in T$ and a point p :

$$((a, b), p) \mapsto p',$$

by some transformation.

The corresponding co-action

An action

$$\mu : \mathbb{G}_m^2 \times_{\mathrm{Spec}(k)} \mathrm{Spec}(R) \to \mathrm{Spec}(R),$$

corresponds to a map (where $\mathcal{O}(T) = k[t_1, t_2, t_1^{-1}, t_2^{-1}]$):

$$\mu' : R \longrightarrow \mathcal{O}(T) \otimes_k R.$$

The torus action can be completely understood through this algebraic viewpoint.

T-equivariant R-modules

A T-**equivariant** R-module M is an ordinary R-module, with a map:

$$\rho : M \longrightarrow \mathcal{O}(T) \otimes_k M,$$

subject to certain axioms. The same module M could have many different $\mathcal{O}(T)$-actions.

T-equivariant R-modules (cont.)

A map $f : (M, \rho) \to (N, \rho')$ is T-**equivariant** if:

$$\begin{array}{ccc} M & \xrightarrow{\quad f \quad} & N \\ \downarrow{\scriptstyle \rho} & & \downarrow{\scriptstyle \rho'} \\ \mathcal{O}(T) \otimes_k M & \xrightarrow{\mathrm{id} \otimes f} & \mathcal{O}(T) \otimes_k N, \end{array}$$

i.e., the map f commutes with the $\mathcal{O}(T)$-action.

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- two objects are generally the same if they have the same Ext-groups.

Important: in this setting, a module M and it's projective resolution $P_\bullet$ are isomorphic.

Some results

For the ring $R := k[x, y]/(xy)$, with $\mathcal{O}(T) = k[t, t^{-1}]$, there are objects $\mathcal{L}_n, \mathcal{R}_n$:

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Bertoloni-Meli demonstrates the following projective resolution for $\mathcal{L}_n$:

$$\cdots \longrightarrow \mathcal{R}_{n+4} \xrightarrow{\cdot y_1} \mathcal{R}_{n+2} \xrightarrow{\cdot x_1} \mathcal{R}_{n+2} \xrightarrow{\cdot y_1} \mathcal{R}_n \xrightarrow{\cdot x_1} \mathcal{R}_n.$$

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This can be used to define more objects ϵ_n and δ_n, and compute extensions between all these objects.

Some results (cont.)

We consider the more general case

$$R = k[x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_{n-1}]/(x_i y_i)_{i=1}^{n-1}.$$

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- we constructed projective resolutions for $\mathcal{L}_\alpha$,
- and calculated Ext-groups between $\mathcal{L}_\alpha, \delta^i_\alpha, \epsilon^i_\alpha$ for $1 \leq i \leq n-1$,
- used these Ext-groups to uniquely characterize certain special objects.

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