On Shalika germs for equivalued elliptic orbits

Leo Hong, Ari Shpilman
Mentored by Thomas Rüd

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Outline

- Motivation: Fourier analysis
- p-adics
- Orbits, Homogeneity, and the Shalika germ expansion
- Cartan Decomposition
- Flag Varieties
- Future Work

Selberg's Trace Formula

- If f is a well behaved function on $\mathbb{R}$, Fourier analysis tells us that we can decompose it into a "sum" of $e^{2\pi ir}$ for $r \in \mathbb{R}$, with coefficients $\hat{f}(r)$.

Poisson Summation Formula

$$\sum_{n\in\mathbb{Z}} f(n) = \sum_{n\in\mathbb{Z}} \hat{f}(n)$$

- We can generalize $\mathbb{R}$ to any well behaved group G and $\mathbb{Z}$ to some discrete subgroup $\Gamma \subset G$.

$$\sum_{\gamma\in\Gamma} \text{weights} \int_{G_\gamma\backslash G} f(g^{-1}\gamma g)dg = \text{spectral terms}$$

- The orbital integral is the trace of an infinite dimensional "matrix", with entries $K(x,y) = \sum_{\gamma\in\Gamma} f(x^{-1}\gamma y)$.

Orbital Integrals

For this talk, we assume G is a p-adic linear algebraic group, such as $\mathrm{SL}_n(\mathbb{Q}_p)$, $\mathrm{GL}_n(\mathbb{Q}_p)$, or $\mathrm{Sp}_{2n}(\mathbb{Q}_p)$. We wish to compute orbital integrals

$$\mathrm{Orb}(X, f) = \int_{G_X \backslash G} f(g^{-1} X g) dg$$

where X is diagonalizable and its characteristic polynomial is irreducible (along with equivalued).

Remark: X is an element of the lie algebra $\mathfrak{g}$ of G, an "additive" version of G. Throughout the talk, G will be $\mathrm{GL}_n(\mathbb{Q}_p)$ and $\mathfrak{g}$ will be $M_n(\mathbb{Q}_p)$.

$\mathbb{Z}_p$ and $\mathbb{Q}_p$

Problem: Want to do Fourier analysis over $\mathbb{Q}$, but $\mathbb{Q}$ has "holes".
Solution: Look at completions of $\mathbb{Q}$.

Ostrowski's Theorem

The completions of $\mathbb{Q}$ are $\mathbb{R}$ and $\mathbb{Q}_p$ for primes p.

$\mathbb{Z}_p$ and $\mathbb{Q}_p$

First, we look at how $\mathbb{R}$ is constructed from $\mathbb{Q}$.

- The "holes" in $\mathbb{Q}$ come from the fact that there are non convergent (bounded) sequences.
- We can define real numbers as equivalence classes of sequences

$$\begin{aligned}\sqrt{2} &= 1.414... \\ &= (1, 1.4, 1.41, 1.414, \dots) \\ &= (1, 1.5, 1.42, 1.415, \dots)\end{aligned}$$

- The rational numbers are the ones that have some representation which is eventually constant.

$\mathbb{Z}_p$ and $\mathbb{Q}_p$

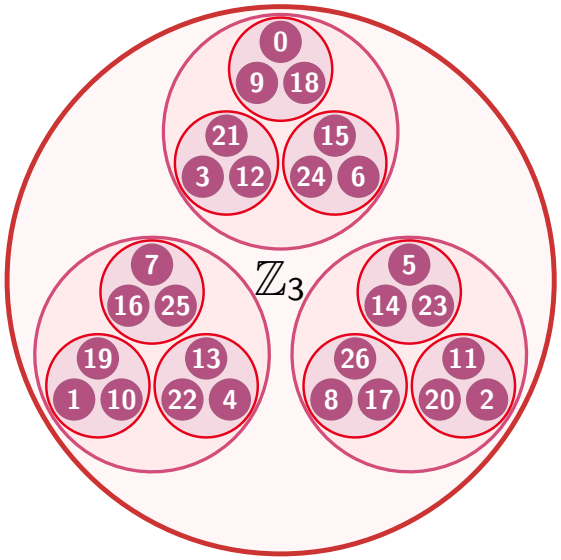
- Fix a prime p. Given a number, we can consider their base p expansion.

$$67 = \overline{\ldots 0124}_7$$

- We can allow numbers with "infinitely many" digits. In terms of sequences,

$$\overline{\ldots a_2 a_1 a_0}_p = (\overline{\ldots 0 a_0}_p, \overline{\ldots 0 a_1 a_0}_p, \overline{\ldots 0 a_2 a_1 a_0}_p)$$

This gives us $\mathbb{Z}_p$, the p-adic integers.



$\mathbb{Z}_p$ and $\mathbb{Q}_p$

- If we allow *finitely* many nonzero digits after the decimal point, we get $\mathbb{Q}_p$, the p-adic rationals. For example, in $\mathbb{Q}_7$,

$$\overline{\ldots 0.6} = \frac{6}{7}.$$

- Note that there are non-integer rational numbers in $\mathbb{Z}_p$. For example, in $\mathbb{Z}_7$,

$$\overline{\ldots 111} = -\frac{1}{6}.$$

Orbits

Definition

Given a matrix $X \in M_n(\mathbb{Q}_p)$, its *orbit* is given by

$$\mathcal{O}_X = \{g^{-1}Xg : g \in \mathrm{GL}_n(\mathbb{Q}_p)\}.$$

The name "*Orbital Integral*" comes from the following:

$$\mathrm{Orb}(X, f) = \int_{G_X \backslash G} f(g^{-1}Xg)dg = \int_{\mathcal{O}_X} f(Y)dY.$$

Example

If $X \in M_n(\mathbb{Q}_p)$ is diagonalizable, then its orbit is exactly the elements with the same eigenvalues and multiplicities.

Definition

A matrix $N \in M_n(\mathbb{Q}_p)$ is nilpotent if $N^k = 0$ for some k.

Orbits

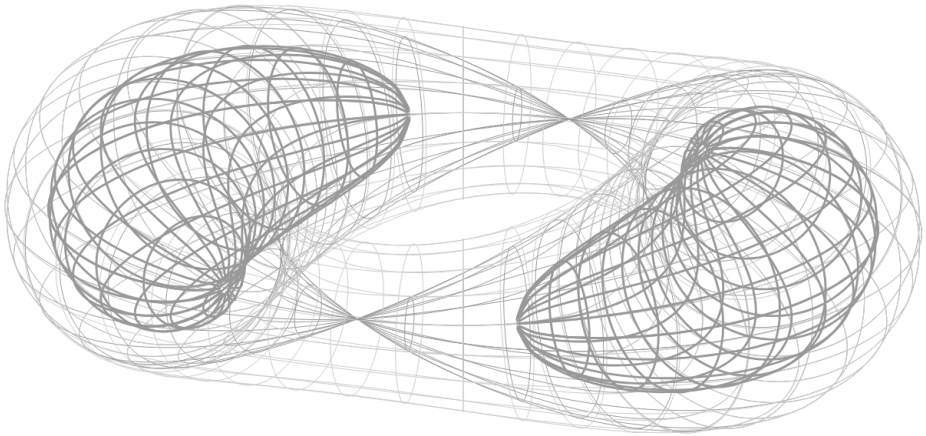


Figure: Orbits of $\mathrm{SL}_2(\mathbb{R})$

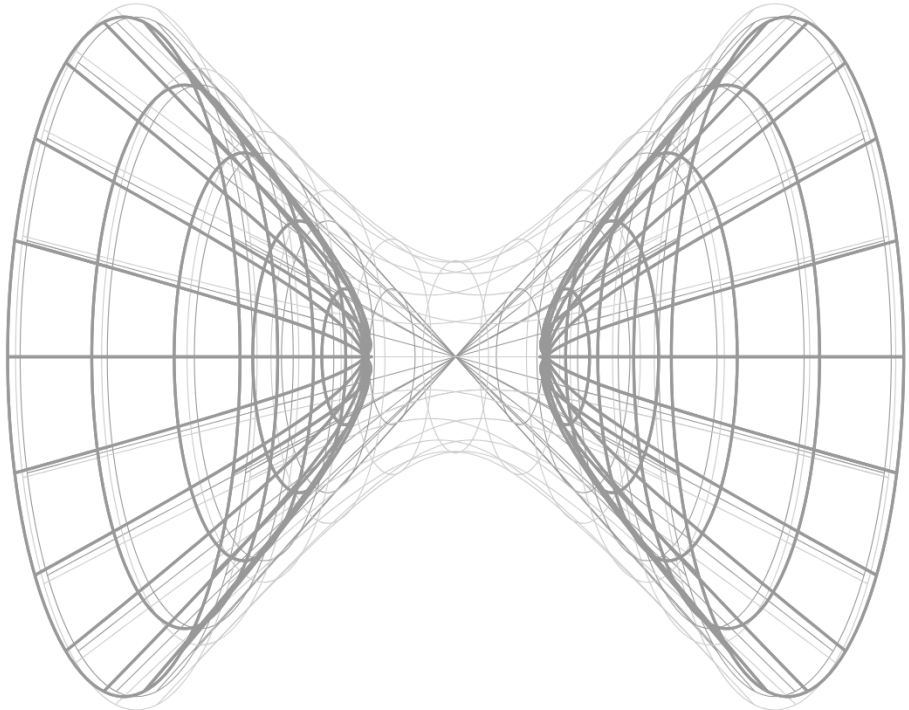


Figure: Orbits of $\mathfrak{sl}_2(\mathbb{R})$, or $\{X \in M_2(\mathbb{R}) : \mathrm{tr}(X) = 0\}$

Orbits

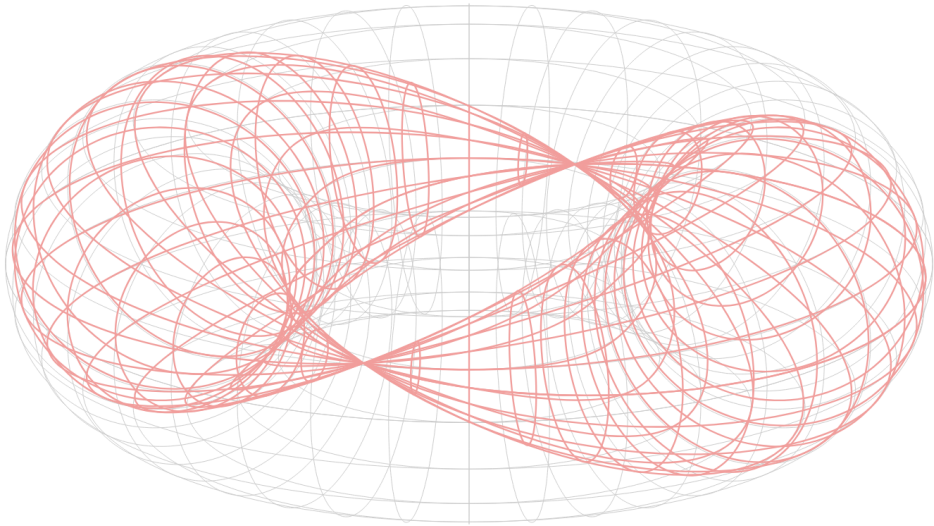


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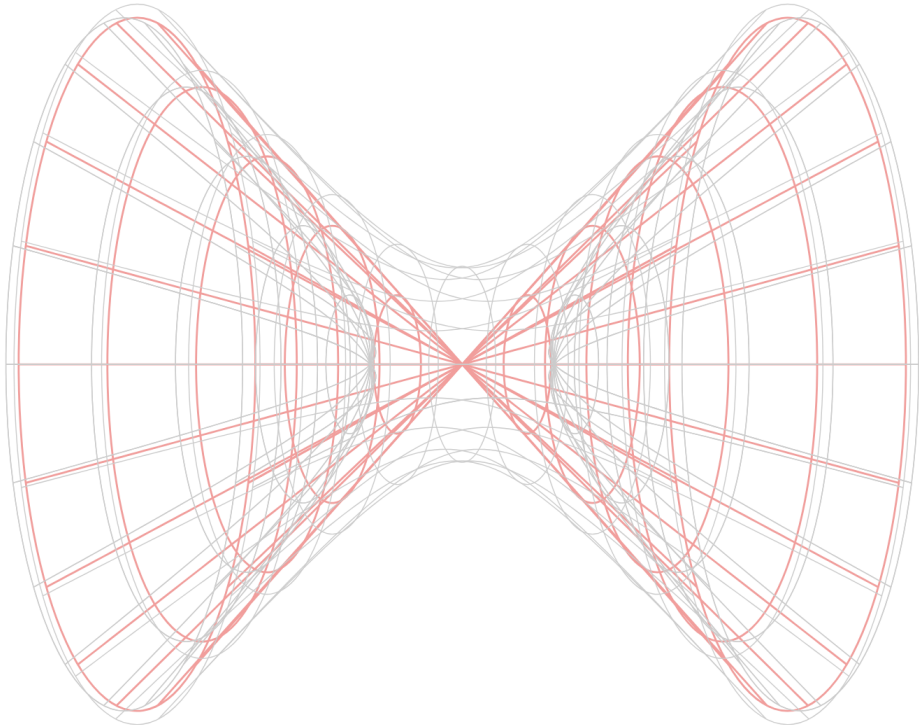
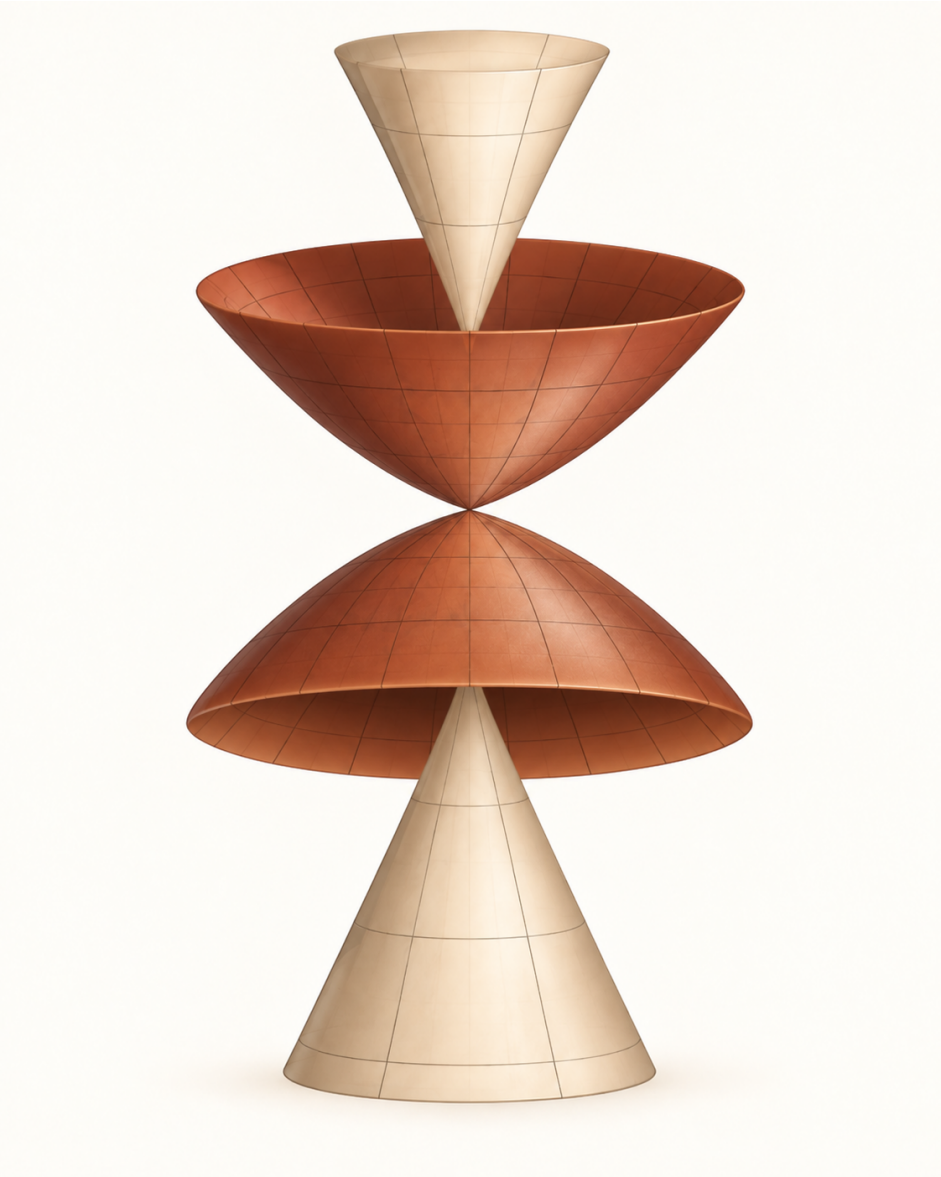


Figure: Orbits of $\mathfrak{sl}_2(\mathbb{R})$, or $\{X \in M_2(\mathbb{R}) : \mathrm{tr}(X) = 0\}$

Orbit Ecosystem - MUSHROOMS!



Shalika Germ Expansion

Number the nilpotent orbits $\mathcal{O}_1, \ldots, \mathcal{O}_r$, and choose representatives $N_i \in \mathcal{O}_i$.

Shalika Germ Expansion

There exists functions $\Gamma_i : \mathfrak{g}_{\mathrm{rs}} \to \mathbb{R}$ such that for every $f \in C_c^\infty(G)$,

$$\mathrm{Orb}(X, f) = \sum_{i=1}^{r} \Gamma_i(X)\mathrm{Orb}(N_i, f)$$

for all $X \in \mathfrak{g}_{\mathrm{rs}}$ close enough to 0.

- The Γ_i are called Shalika germs.
- Because the Shalika germs don't depend on f, (regular semsimple) orbits can be "decomposed" into nilpotent orbits.

Nilpotent Orbits

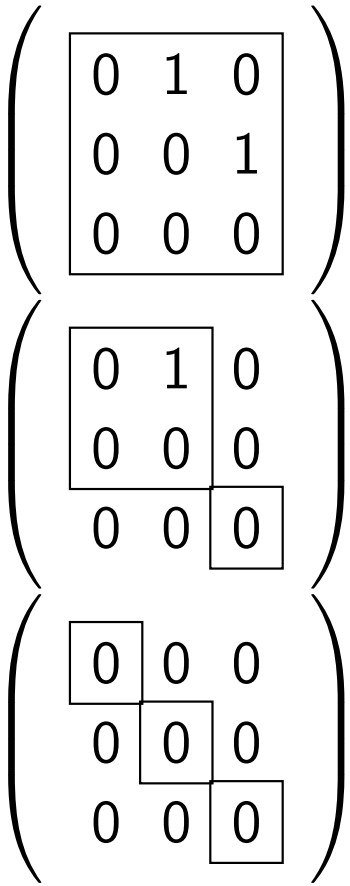
- Each nilpotent element of $M_n(\mathbb{Q}_p)$ has a Jordan canonical form with Jordan blocks of size $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k$ for some k.
- Therefore nilpotent orbits $\mathcal{O}_\lambda$ are indexed by partitions λ of n. $\mathcal{O}_\lambda$ consists of the nilpotent matrices with Jordan blocks of size $\lambda_1, \lambda_2, \ldots, \lambda_k$.

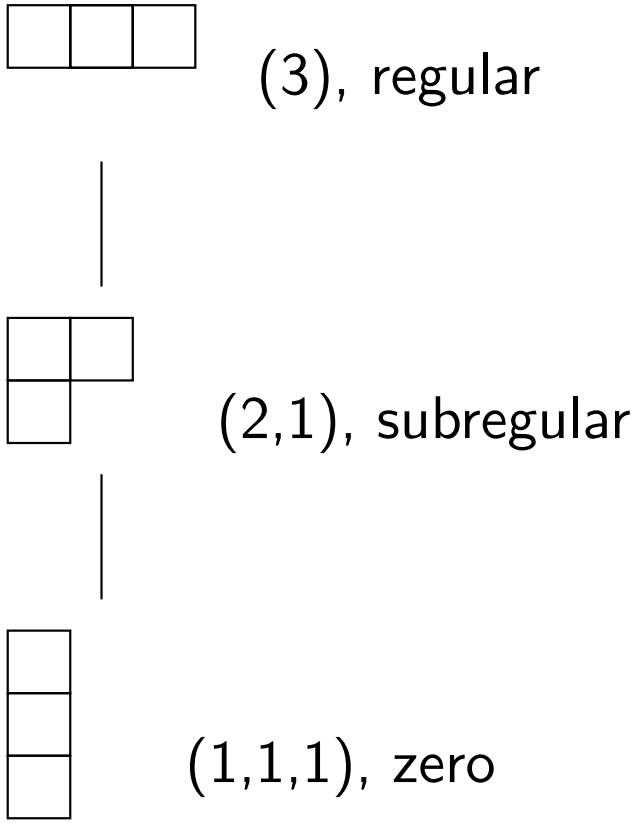
$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Figure: Matrix corresponding to the partition $(3, 2, 2)$ of 7

Dominance ordering

- The closure $\overline{\mathcal{O}}$ consists only of nilpotent elements.
- Let $\mathcal{O} \geq \mathcal{O}'$ if $\mathcal{O}' \subseteq \overline{\mathcal{O}}$. In particular, $\dim(\mathcal{O}) \geq \dim(\mathcal{O}')$.

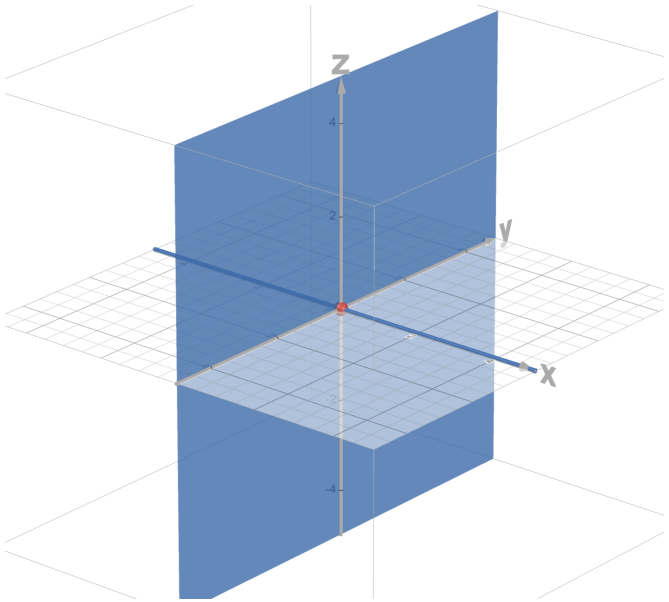




Dominance ordering

Upper triangular slice of $M_3(\mathbb{Q}_p)$:

$$\begin{pmatrix} 0 & x & y \\ 0 & 0 & z \\ 0 & 0 & 0 \end{pmatrix}$$



- (0 orbit) When $\lambda = (1, 1, 1)$, $\mathcal{O}_\lambda = \{0\}$.
- (Subregular orbit) When $\lambda = (2, 1)$, $x = 0$ or $y = z = 0$, but not both.
- (Regular orbit) $\lambda = (3)$ consists of everything else.

Homogeneity

Nilpotent orbits $\mathcal{O}$ have the special property that if $N \in \mathcal{O}$, then $\alpha N \in \mathcal{O}$ for all $\alpha \in \mathbb{Q}_p$. In particular, we can let α be a power of p.

Theorem

Given a nilpotent element N_i,

$$Orb(p^{2n} N_i, f) = p^{\dim(\mathcal{O}_N)n} \mathrm{Orb}(N_i, f)$$

Applying this to the Shalika Germ Expansion, we get

$$\mathrm{Orb}(p^{2n} X, f) = \sum_{i=1}^{r} p^{\dim(\mathcal{O}_i)n} \Gamma_i(X) \mathrm{Orb}(N_i, f)$$

As $n \to \infty$, the terms $\Gamma_i(X)\mathrm{Orb}(N_i, f)$ separate.

Triangular Test Functions

Theorem (DeBacker 2002)

For every nilpotent orbit $\mathcal{O}_i$, there exists a function $f_{\mathcal{O}_i} \in C_c^\infty(G)$ such that $\mathrm{Orb}(N_j, f_{\mathcal{O}_i}) \neq 0$ only if $\mathcal{O}_j \geq \mathcal{O}_i$. In particular, $\mathrm{Orb}(N_j, \mathcal{O}_i) \neq 0$ only if

$$\dim(\mathcal{O}_j) \leq \dim(f_{\mathcal{O}_i}),$$

with equality iff $i = j$.

This means

$$\mathrm{Orb}(p^{2n}X, f_{\mathcal{O}_i}) = p^{\dim(\mathcal{O}_i)n}\Gamma_i(X)\mathrm{Orb}(N_i, f_{\mathcal{O}_i}) + \text{higher order terms.}$$

To compute $\Gamma_i(X)$, we just need to compute the lowest order terms of $\mathrm{Orb}(p^{2n}X, f_{\mathcal{O}_i})$ as $n \to \infty$. In particular, we can write it in base p and look at the last digits.

p-Adic Integrals and Cell Decomposition

Let $G = \mathrm{GL}_1(\mathbb{Q}_p) = (\mathbb{Q}_p \setminus \{0\}, \times)$. Consider the integral

$$\int_{G_x \setminus G} \mathbb{1}_{\mathbb{Z}_p \setminus p^{1434}\mathbb{Z}_p}(g^{67}) dg$$

In other words, we want to find $\mathrm{vol}(S)$ where

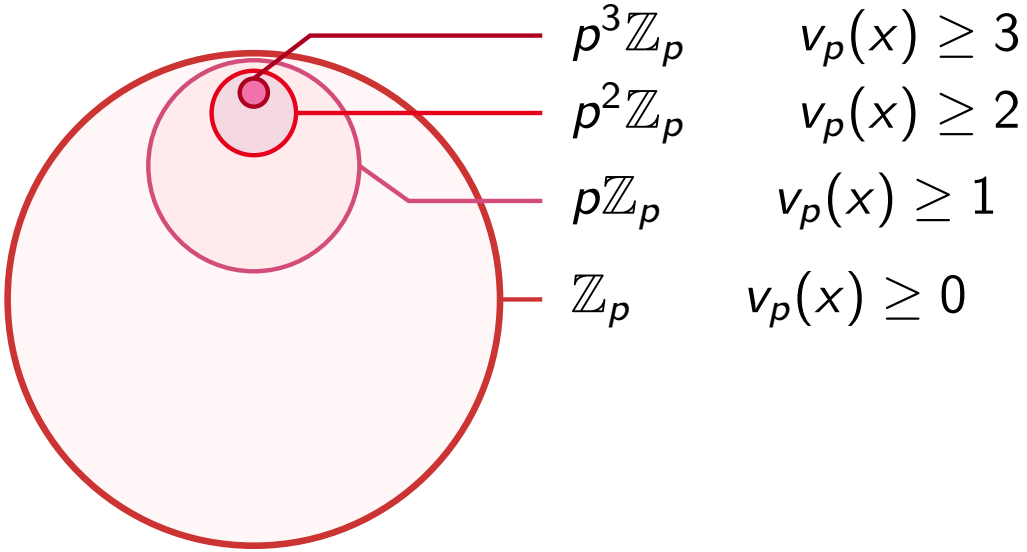
$$S = \{g \in G : 0 \leq v_p(g^{67}) < 1434\}.$$

- Let $K = \mathrm{GL}_1(\mathbb{Z}_p) = \mathbb{Z}_p \setminus p\mathbb{Z}_p$. Then

$$G = \bigsqcup_{n \in \mathbb{Z}} Kp^n,$$

so it suffices to find $\mathrm{vol}(S_n)$ for all n, where $S_n = S \cap Kp^n$.

p-Adic Integrals and Cell Decomposition



Each layer between successive circles represents Kp^n for some n, and has a fixed p-adic valuation of n.

- $S_n = Kp^n = p^n\mathbb{Z}_p \setminus p^{n+1}\mathbb{Z}_p$ if $0 \leq n \leq 21$, and $S_n = \varnothing$ otherwise.
- With a standard Haar measure of $\mathrm{vol}(\mathbb{Z}_p^\times) = 1$, each layer has volume 1, so our integral is

$$\sum_{n=0}^{21} 1 = \boxed{22}.$$

Cartan Decomposition

In general, we can split up G into cells based on a generalized "p-adic valuation". Let $G = \mathrm{GL}_n(\mathbb{Q}_p)$ and $K = \mathrm{GL}_n(\mathbb{Z}_p)$. Then

$$G = \bigsqcup_{\lambda_1 \geq \cdots \geq \lambda_n} K p^{\lambda_1, \ldots, \lambda_n} K$$

where

$$p^{\lambda_1, \ldots, \lambda_n} = \begin{pmatrix} p^{\lambda_1} & 0 & \cdots & 0 \\ 0 & p^{\lambda_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & p^{\lambda_n} \end{pmatrix}$$

Cartan Decomposition

Consider the case where $G = \mathrm{GL}_3(\mathbb{Q}_p)$. We can compute

$$\mathrm{Orb}(p^n X, f) = \int_{G_x \backslash G} \mathbb{1}_{M_3(\mathbb{Z}_p)}(g^{-1} p^n X g) dg$$

by summing

$$\int_{K p^{\lambda_1, \lambda_2, \lambda_3} K} \mathbb{1}_{M_3(\mathbb{Z}_p)}(g^{-1} p^n X g) dg$$

for $\lambda_1 \geq \lambda_2 \geq \lambda_3$. We only care about $\lambda_1 - \lambda_2 = a$ and $\lambda_2 - \lambda_3 = b$.

Recall that GL_3 has 3 nilpotent orbits, and therefore 3 shalika germs.

- Regular orbit- (3)
- Subregular orbit- $(2, 1)$
- 0 orbit- $(1, 1, 1)$

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References I



DeBacker, Stephen (2002). "Parametrizing nilpotent orbits via Bruhat-Tits theory". In: *Ann. of Math. (2)* 156.1, pp. 295–332. ISSN: 0003-486X,1939-8980. DOI: 10.2307/3597191. URL: https://doi.org/10.2307/3597191.