A Representation-Theoretic Computation of the Alexander Polynomial of Satellite Knots

Wesley Lin
Mentor: Kenta Suzuki

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1 Satellite Knots and the Alexander Polynomial

2 Satellite Knot Construction

3 Coloring Strands with $U_q(\mathfrak{sl}(1|1))$-modules

4 Main Result

5 Conclusion

Knots and Links

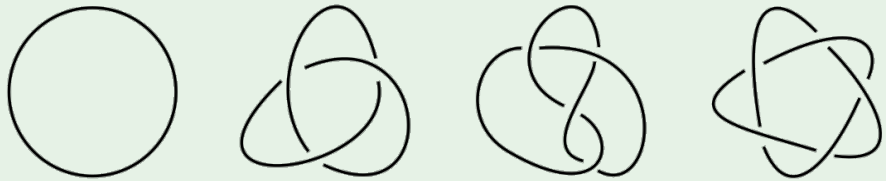
Definitions

- A **knot** is an embedding of the circle $\mathbb{S}^1$ in $\mathbb{R}^3$.
- A **link** is an embedding of several circles in $\mathbb{R}^3$.

Link diagrams project knots and links onto two-dimensional space.

Examples

- Some knots:



Unknot (0_1), **trefoil (3_1)**, **figure-eight (4_1)**, and **cinquefoil (5_1)**.

- Some links:



Hopf link (2_1^2), **Whitehead link (5_1^2)**, and **Borromean link (6_2^3)**.

Slide adapted from Eddy Li, 2025

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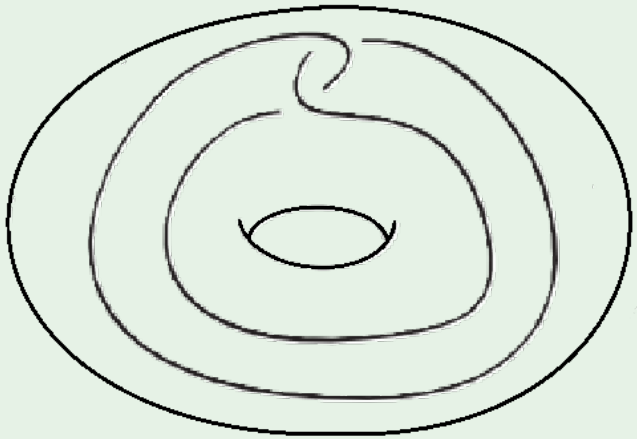
The Winding Number

Let P be a knot in the solid torus $\mathbb{S}^1 \times \mathbb{D}^2$. Give the core of the solid torus an orientation (clockwise or counterclockwise).

Definition

The **winding number** of P is the number of times P wraps around the center of the torus, taking direction into account.

Examples



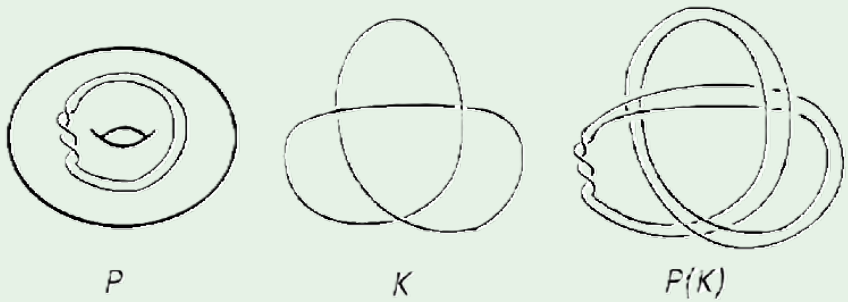
The **Whitehead double** has a winding number of 0.

Satellite Knots

Definition

- Let K be a knot in $\mathbb{R}^3$ and let P be a knot in the solid torus. The **satellite knot** $P(K)$ is the image of P after gluing the solid torus into a tubular neighborhood of K.
- $P(K)$ is said to be the satellite knot with **companion** K and **pattern** P.

Example



A satellite knot where the companion is the trefoil.

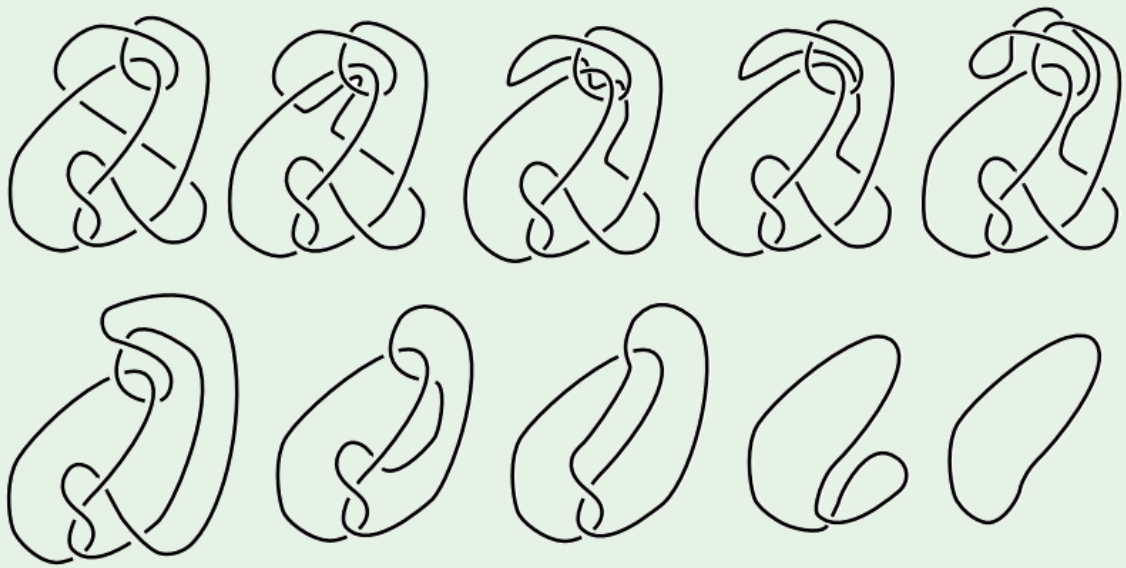
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Link Invariants

Question

When do two link diagrams represent the same link?

Example



Sometimes, links that look complicated are actually isotopic to simple links.

Link invariants are quantities associated with a link independent of a choice of diagram. They help answer this question.

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The Alexander Polynomial

Definition

Let L_+, L_-, and L_0 be three links that differ only at one crossing.



The **Alexander polynomial** $\Delta_L(q^2) \in \mathbb{C}(q)$ is defined by the recursive skein relation

$$\Delta_{L_+}(q^2) - \Delta_{L_-}(q^2) = (q - q^{-1})\Delta_{L_0}(q^2).$$

We let the Alexander polynomial of the unknot (0_1) is 1.

Theorem (Alexander, 1920)

The Alexander polynomial is a link invariant.

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The Alexander Polynomial of Satellite Knots

Seifert computed a formula for the Alexander polynomial of the satellite knot $P(K)$.

Theorem (Seifert, 1950)

Let $P(K)$ be a satellite knot with companion K and pattern P. Then

$$\Delta_{P(K)}(q^2) = \Delta_P(q^2) \cdot \Delta_K(q^{2w})$$

where w is the winding number of P inside the solid torus.

This formula is unclear from the skein relation definition of the Alexander polynomial.

Motivating Question

- Can we use algebra to understand Seifert's satellite knot formula?
- Specifically, where does this mysterious winding number w in the exponent come from?

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Tangles

Definition

- An **oriented tangle** is a set of oriented strands in $\mathbb{R}^2 \times [0,1]$ with endpoints in $\mathbb{R}^2 \times \{0,1\}$.
- Tangle diagrams (two-dimensional projections of tangles) are generated by crossings, caps, and cups:

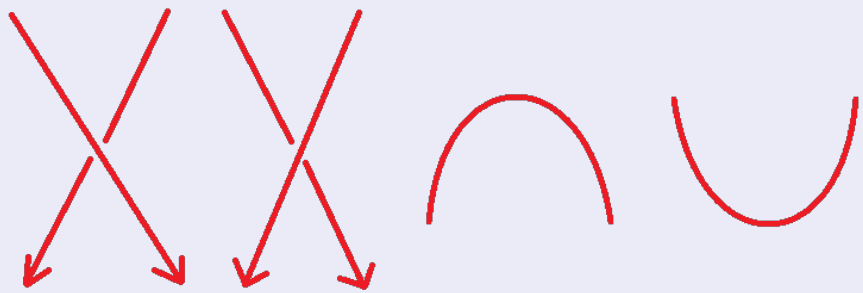
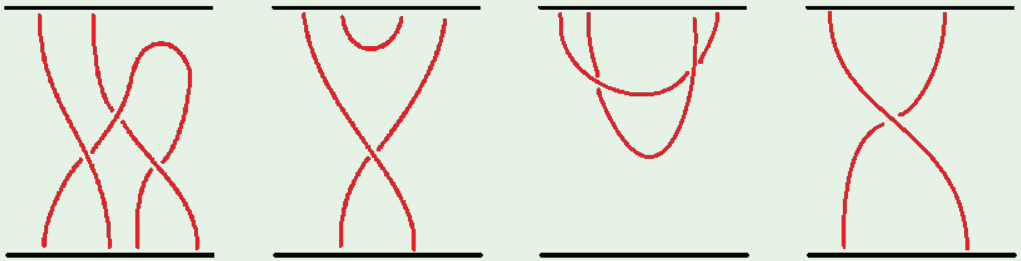


Figure: A positive crossing, a negative crossing, a cap, and a cup

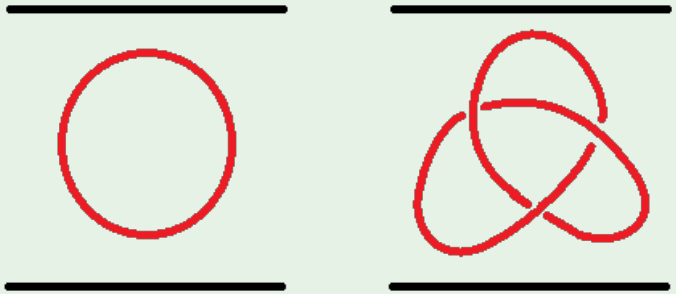
Tangles

Examples

The following are tangles:

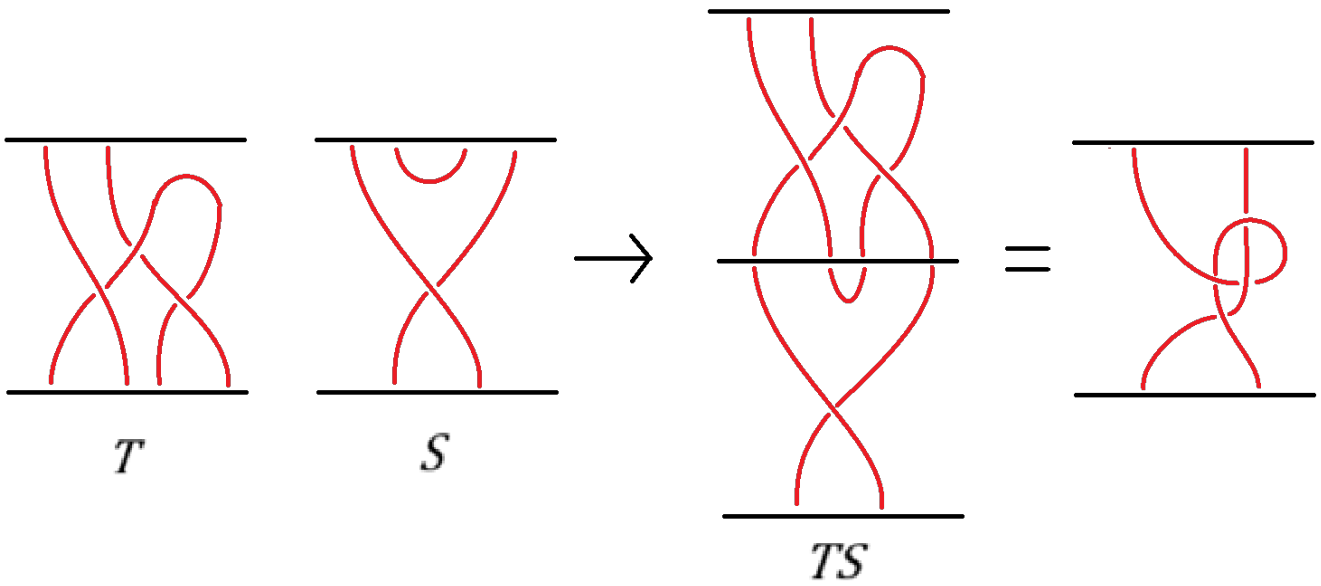


Links are tangles with no endpoints.



Tangles

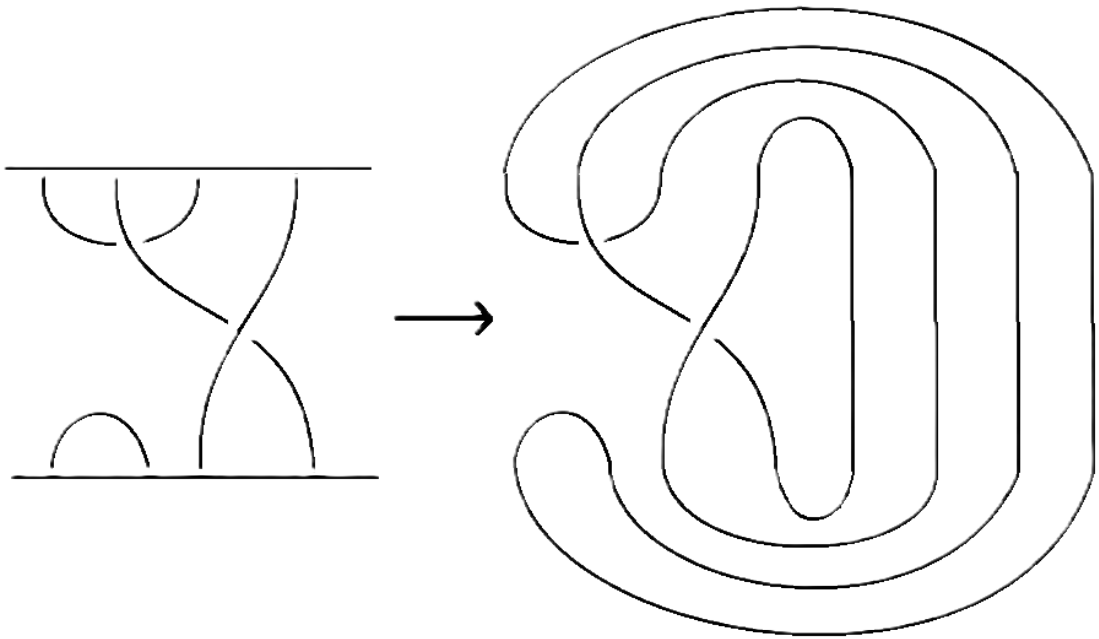
We define multiplication of tangles T and S by attaching T on top of S (when the orientations of T and S agree).



Tangles

Definition

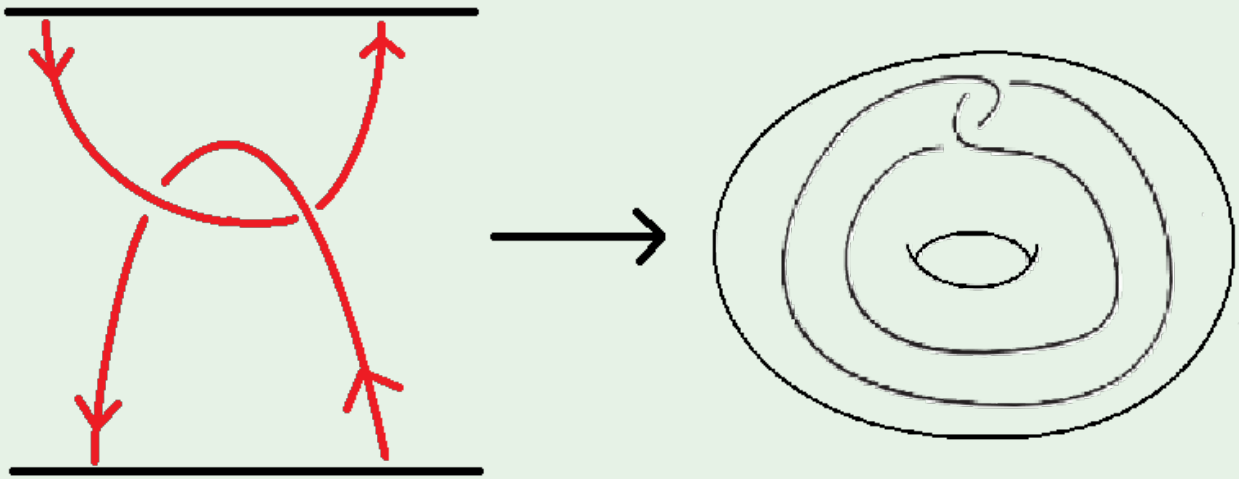
Then the **closure** of T is the link $\hat{T}$ constructed by connecting the top and bottom endpoints of T.



Tangles

Remark

Let T have a strands oriented downward and b strands oriented upward at both the top and bottom endpoints. The winding number of the closure of T in the solid torus is $a - b$.

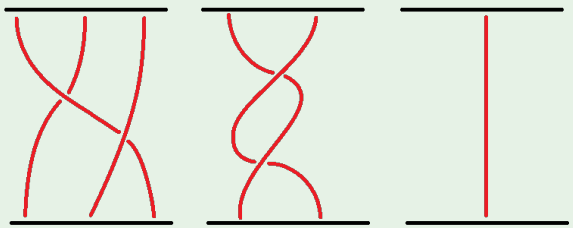


Braids

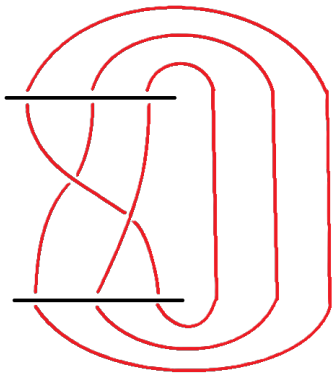
A **braid** is a special type of tangle generated only by crossings that is particularly easy to understand.The set of braids with n strands forms the braid group B_n.

Examples

Some braids:



Since all braids in B_n have n top and bottom endpoints, the closure operation is defined for all braids.



Constructing the Satellite Knot

Definition

Let $\delta_k : B_n \to B_{kn}$ be the **cabling homomorphism** that duplicates each strand in a braid into k parallel strands.

Example

Red strands cross over blue strands in the following example.

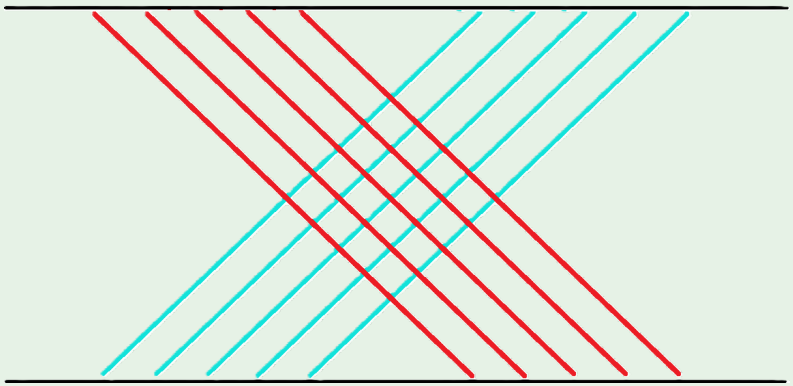


Figure: A crossing under δ_5

Constructing the Satellite Knot

- Let the companion $K \in \mathbb{R}^3$ be the closure of $\beta \in B_n$.
- Let the pattern $P \in \mathbb{S}^1 \times \mathbb{D}^2$ be the closure of an oriented tangle T with k downward-oriented endpoints.

Proposition

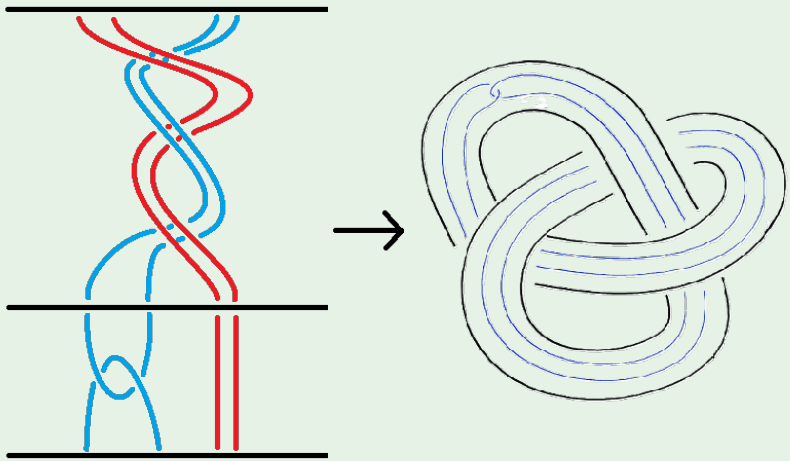
The satellite knot $P(K)$ is the closure of $\delta_k(\beta)T$ where T is a tangle on the first k strands.

Abuse of Notation

- T is technically a tangle on k endpoints, but the tangle $\delta_k(\beta)T$ has nk endpoints.
- T is applied on the first k strands while the rest are unaffected (attach the trivial tangle).

Constructing the Satellite Knot

Example



Here, the closure of β is the trefoil, the closure of T is the Whitehead double, and T has one upward- and one downward-oriented endpoint.

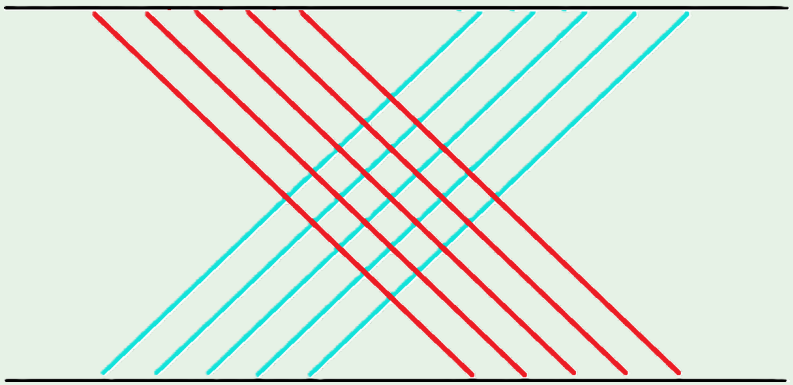
Remark

We focus only on the case when all strands of T are oriented downward.

Constructing the Satellite Knot

The image of braid generators under δ_k is quite unwieldy.

Example (revisited)



One crossing becomes k^2 crossings under δ_k.

Question

Can we somehow represent k parallel strands with one "thick" strand?

Coloring Strands

Modules of the quantum enveloping superalgebra $U_q(\mathfrak{sl}(1|1))$ (vector spaces over $\mathbb{C}(q)$ with a left $U_q(\mathfrak{sl}(1|1))$-action) carry the action of tangles.

- Let M be a $U_q(\mathfrak{sl}(1|1))$-module.
- Let a tangle T be downward-oriented with k endpoints.
- We can "color" strands with M.
- Then T acts as an endomorphism of $M^{\otimes k}$.

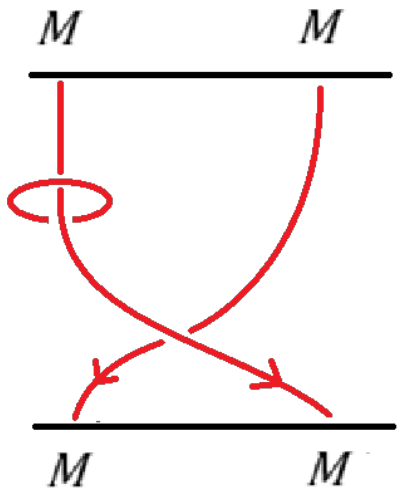
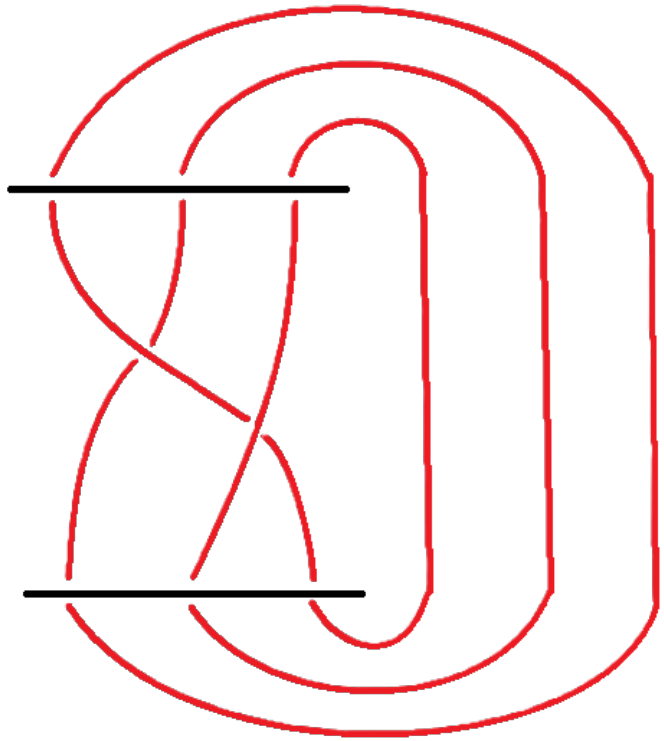


Figure: An endomorphism of $M \otimes M$.

The Trace

Notice that closing strands of a tangle T simply adds caps and cups above and below T.



Algebraically, this construction is almost identical to the trace in linear algebra.

The Trace

Definition

Let V be a vector space over $\mathbb{C}(q)$. The **trace** tr is a map from $\mathrm{End}(V)$ to $\mathbb{C}(q)$.

Let's take inspiration from this definition and modify the trace into

$$\tilde{\mathrm{tr}}\colon\ \mathrm{End}(M^{\otimes k}) \to \mathbb{C}(q)$$

to give us a scalar associated with the action of T on $M^{\otimes k}$. Specifically, taking the trace of a tangle T corresponds to closing it.

Coloring Strands

Simple $U_q(\mathfrak{sl}(1|1))$-modules $L(x)$ are especially important to us.

Theorem

Color the strands of T with a simple $U_q(\mathfrak{sl}(1|1))$-module $L(x)$ where $x \in \mathbb{Z}$. Then

$$\tilde{\mathrm{tr}}_{L(x)^{\otimes k}}(T) \doteq \Delta_{\hat{T}}(q^{2x}).$$

Corollary (Sartori, 2015)

Let a tangle T with k endpoints act on the tensor product $L(1)^{\otimes k}$. Then

$$\tilde{\mathrm{tr}}_{L(1)^{\otimes k}}(T) = \Delta_{\hat{T}}(q^2).$$

- We color strands of $\delta_k(\beta)T$ with $L(1)$ to obtain $\Delta_{P(K)}(q^2)$.
- Maybe we can use this to get $\Delta_K(q^{2w})$ in the satellite formula!

Coloring the Satellite Tangle

Notation

Let $W := L(1)^{\otimes k}$.

We color strands with $L(1)$, so T acts on $W = L(1)^{\otimes k}$ and $\delta_k(\beta)$ acts on $L(1)^{\otimes nk} = W^{\otimes n}$.

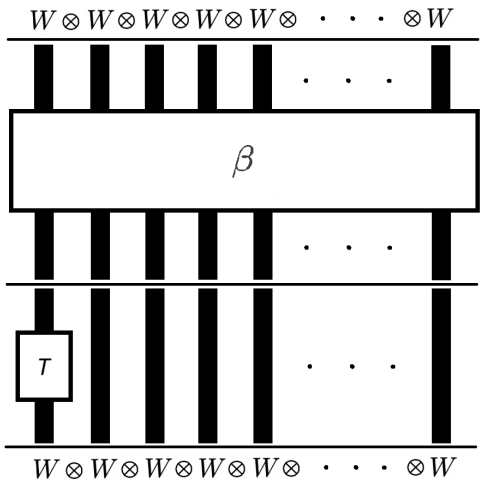


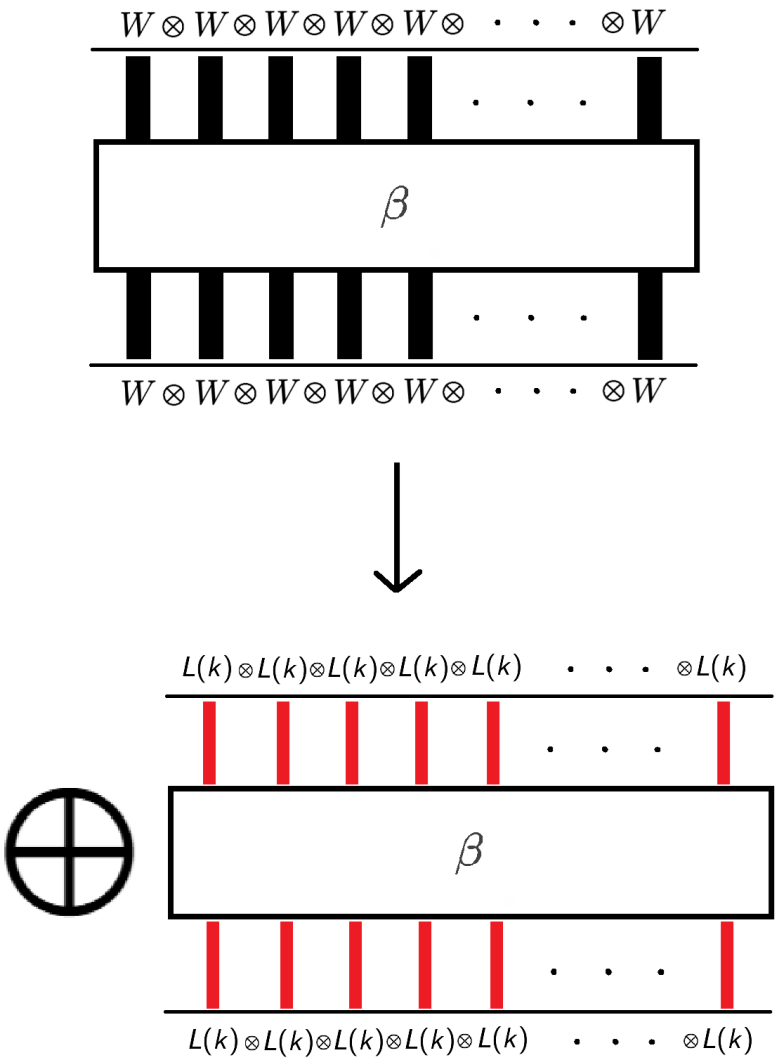
Figure: The tangle $\delta_k(\beta)T$ acting on $W^{\otimes n} = L(1)^{\otimes nk}$.

We have successfully reduced k parallel strands into one strand colored by W!

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A Decomposition of $W^{\otimes n}$

But we can do better. Let's zoom in on the top half (β acting on $W^{\otimes n}$).



$W^{\otimes n}$ is a direct sum of $L(k)^{\otimes n}$. And $L(k)$ is simple!

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Putting it All Together

Theorem (Revisited)

Let S be a downward-oriented tangle with e endpoints. Then

- $\tilde{\mathrm{tr}}_{L(1)^{\otimes e}}(S) = \Delta_{\hat{S}}(q^2)$.
- $\tilde{\mathrm{tr}}_{L(x)^{\otimes e}}(S) \doteq \Delta_{\hat{S}}(q^{2x})$.

We can take the trace of $\delta_k(\beta)T$ to recover the Alexander polynomial. Using what we know, we get:

$$\tilde{\mathrm{tr}}_{L(1)^{\otimes nk}}(\delta_k(\beta)T) = \tilde{\mathrm{tr}}_{L(1)^{\otimes k}}(T) \cdot \left(\sum \tilde{\mathrm{tr}}_{L(k)^{\otimes n}}(\beta)\right).$$

Main Theorem

Let $P(K)$ be the satellite knot with pattern P and companion K. Let w be the winding number of P. Then

$$\Delta_{P(K)}(q^2) = \Delta_P(q^2) \cdot \Delta_K(q^{2w}).$$

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Acknowledgments

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- We thank Dr. Thomas Rued for providing detailed and insightful feedback on the interim report.
- We also extend our deepest gratitude to the MIT PRIMES program and all its organizers for providing the opportunity to conduct meaningful, thought-provoking, and fascinating mathematical research.

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THANK YOU!

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