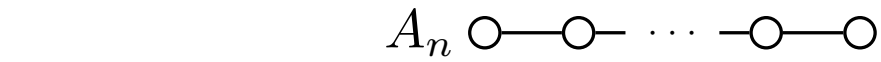
A Closed Formula for Eigenweights

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MIT PRIMES-USA

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Eigenweights



Eigenweights



$$\downarrow$$

$$\epsilon_{\mu,j} = \frac{(n-j)!\,(M_r+1)!}{(n+j+1)!} \prod_{i=0}^{r-1} \frac{i!}{(n'-r+i)!} \cdot K_r, \qquad \text{where}$$

$$K_r = M_r^2\, w(r)^2 + j^2(j+1)^2 \sum_{i=1}^{n} \frac{\min(r,i)\,(n'-\max(r,i))}{n'}\, w(i)^2,$$

$$w(r) = \sum_{i=0}^{j-1} (-1)^i\, \frac{(n-1-i)!\,(i+j+1)!}{(n-j)!\,(i+1)!\,(j-1-i)!\,j(j+1)} \binom{r-1}{i}.$$

Hasse–Weil zeta function

Riemann zeta function,
built from natural numbers:

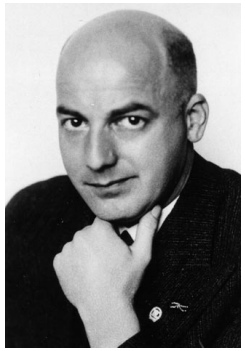
$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$



Riemann

Hasse–Weil zeta function,
built from a curve C over $\mathbb{F}_q$:

$$\zeta_C(s) = \sum_{D \in \mathrm{Div}_{\geq 0}\, C} \frac{1}{(q^{\deg(D)})^s}$$



Hasse



Weil

Shtukas

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- Russian for thing
- Geometric objects built from a curve C over $\mathbb{F}_q$ and a group G.
- The space of all Shtukas has a volume we can measure!

Arithmetic volume of Shtukas

Theorem (Feng–Yun–Zhang, 2026)

For a split semisimple group G, a minuscule coweight μ, and a curve C over $\mathbb{F}_q$, we have

$$\mathrm{Vol}(\mathrm{Sht}^r_{G,\mu}) = q^{(g-1)\dim G}\,(-\log q)^{-r} \cdot \left(\frac{d}{ds}\right)^r\Big|_{s=0} \left(q^{-(2g-2)\,b_\mu\, s} \prod_{j=1}^{n} \zeta_C(\boldsymbol{\epsilon_{\mu,j}}\, s + d_j) \right).$$

- g is the genus of C.
- n is the rank of G.
- The $\boldsymbol{\epsilon_{\mu,j}}$ are called eigenweights. They depend on G, μ, and j.

Lie groups

Wang–Wei (2026) computed eigenweights using Lie algebras and their representations.

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Definition

A **Lie group** is a smooth space (a manifold) with smooth multiplication and inverse operations.

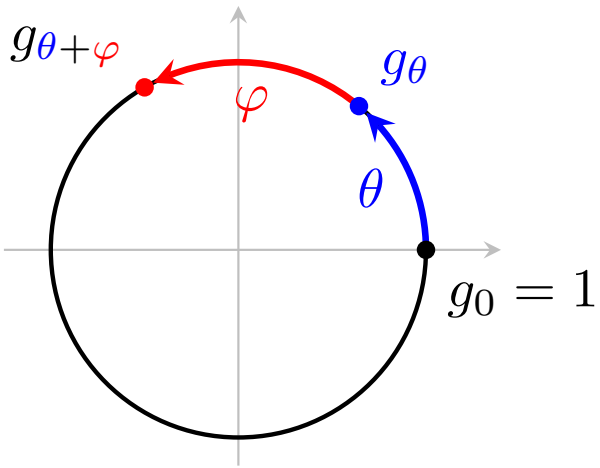
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Example



- Lie group: the rotations of the plane.
- Group operation: $g_\varphi \, g_\theta = g_{\theta+\varphi}$.

Lie algebras

Definition

A **Lie algebra** is the tangent space at the identity of a Lie group.

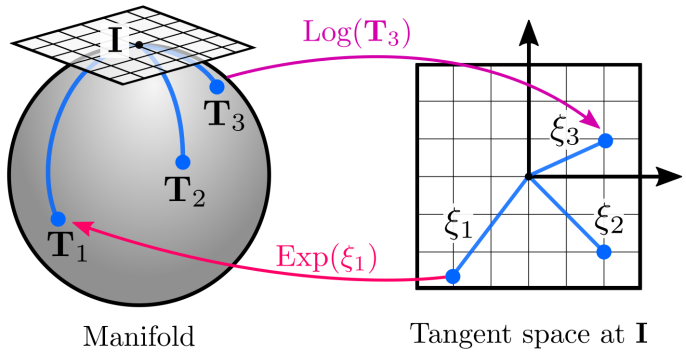


Figure: $\mathrm{SO}(3) \leftrightarrow \mathfrak{so}(3)$ [Mat21]

Lie algebra classifications

We're only concerned with complex, simple Lie algebras.
These Lie algebras have been classified into 9 types (Killing & Cartan).
There are four infinite types and five exceptional types:

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- Type A_n ($\mathfrak{sl}_{n+1}$)
- Type B_n ($\mathfrak{so}_{2n+1}$)
- Type C_n ($\mathfrak{sp}_{2n}$)
- Type D_n ($\mathfrak{so}_{2n}$)
- Type G_2 (14)
- Type F_4 (52)
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We'll use Type A_n $(\mathfrak{sl}_{n+1})$ for now. Let $n' = n + 1$.

$$\mathrm{SL}_{n'} = \{g \in M_{n' \times n'}(\mathbb{C}) \mid \det(g) = 1\}$$

$$\mathfrak{sl}_{n'} = \{X \in M_{n' \times n'}(\mathbb{C}) \mid \mathrm{tr}(X) = 0\}$$

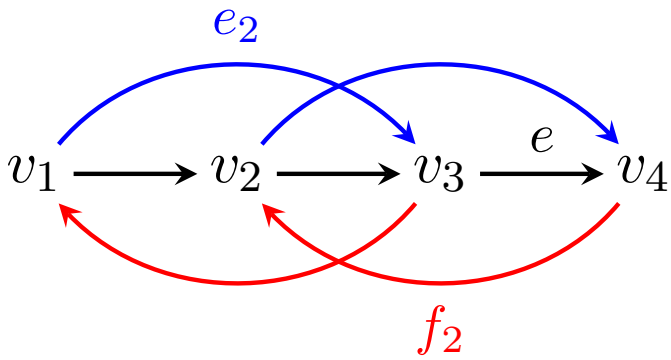
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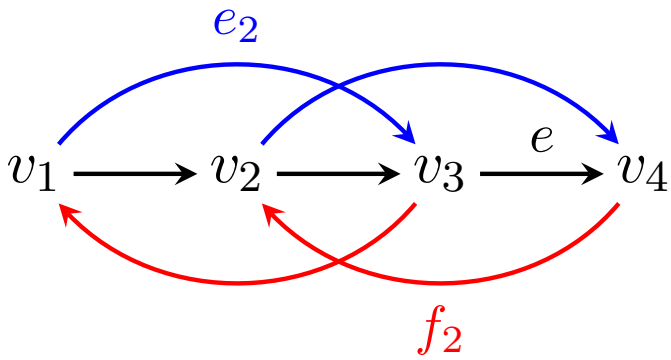
- e: $v_i \mapsto v_{i+1}$ (one step up)
- e_j: $v_i \mapsto v_{i+j}$ (j steps up)
- f_j: $v_i \mapsto v_{i-j}$ (j steps down)



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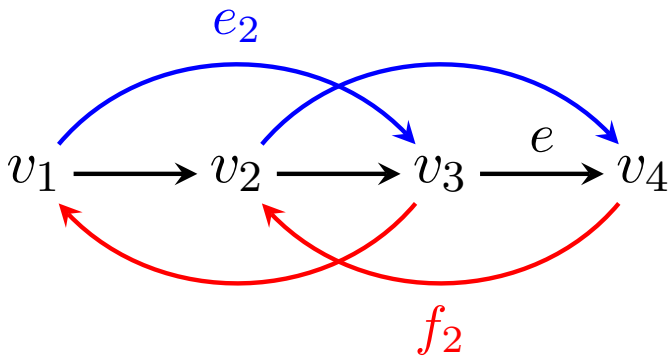


$$e = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

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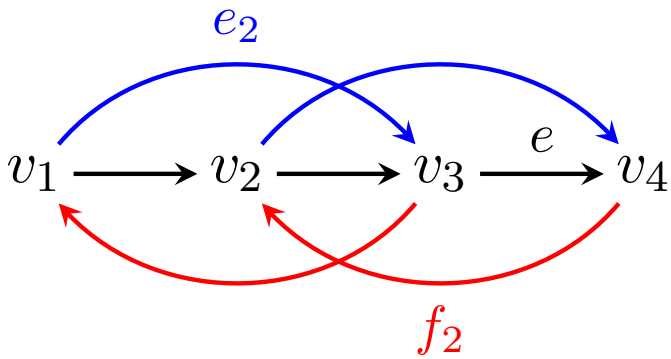
$$\mu = (1^r, 0^{n'-r}) \longrightarrow \bigwedge^r \mathbb{C}^{n'}$$

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In $\bigwedge^r \mathbb{C}^{n'}$, we move one factor at a time (product rule):

$$e(v_1 \wedge v_2) = \underbrace{v_2 \wedge v_2}_{=0} + v_1 \wedge v_3 = v_1 \wedge v_3.$$

Eigenweights

Proposition (Wang–Wei, 2026)

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$u_- = v_1 \wedge \cdots \wedge v_r$, and $u_+ = v_{n'-r+1} \wedge \cdots \wedge v_{n'}$. The operator e goes from u_- to u_+ in $M_r = r(n' - r)$ steps.

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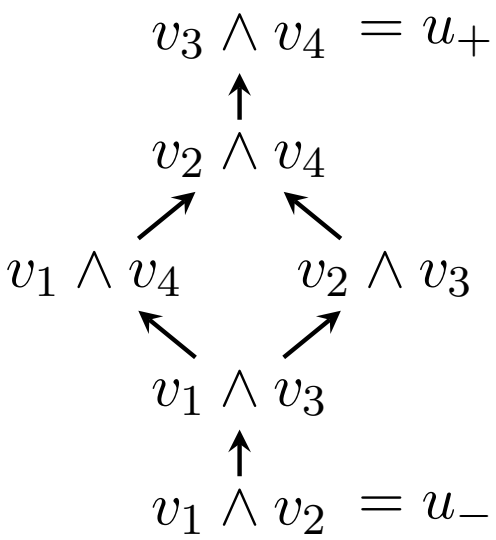
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One term of $\kappa_\mu(e_j, f_j)$:

$$u_- = v_1 \xrightarrow{e^{M_r - s}} v_{n'-s} \xrightarrow{f_j} v_{n'-s-j} \xrightarrow{e_j} v_{n'-s} \xrightarrow{e^s} v_{n'} = u_+$$

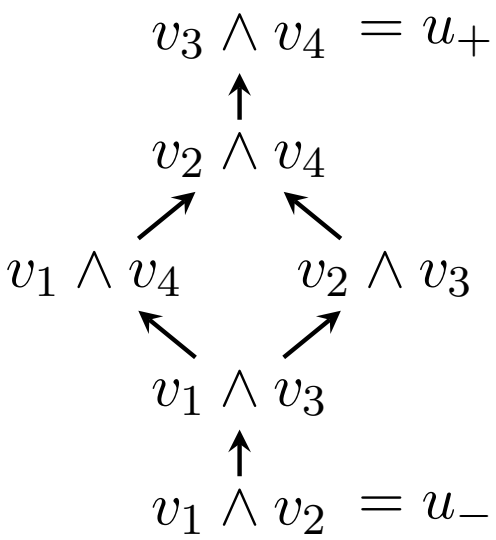
$\bigwedge^r$ Std representation



$\bigwedge^2 \mathbb{C}^4$: using only e

- e can move factors in any order.
- e_j and f_j don't have to move the same factor.
- jumping over a factor flips the sign.

$\bigwedge^r$ Std representation

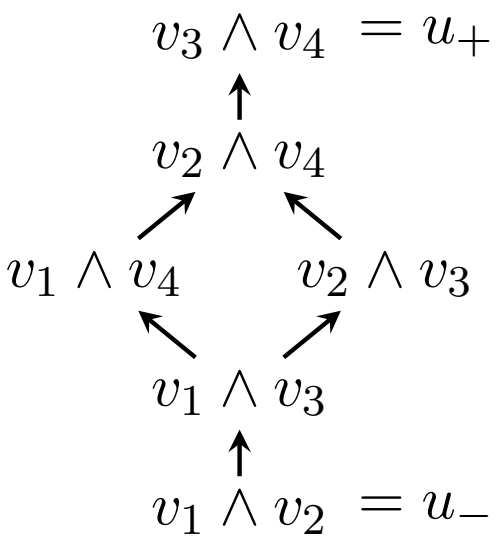


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$\mu \backslash j$	1	2	3	4	5
$\bigwedge^2 \mathbb{C}^4$	4	2	4		
$\bigwedge^2 \mathbb{C}^5$	14	6	9	13	
$\bigwedge^3 \mathbb{C}^6$	198	54	98	110	170

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$\bigwedge^8 \mathbb{C}^{16}$, $j = 1$:

1,485,946,646,567,707,720,957,749,412,147,123,200

(37 digits)

Other formulas

Feng (2026) found a formula for eigenweights in type A_n:

$$\epsilon_{\mu,j} = \sum_{i=0}^{\min(j,r-1)} (-1)^i \chi_{(j,1^{Mr+1})}^{(j+1-i,1^i)+(n'-r)^r}.$$

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Wang–Wei (2026) expanded $\kappa_\mu(e_j, f_j)$ directly to get

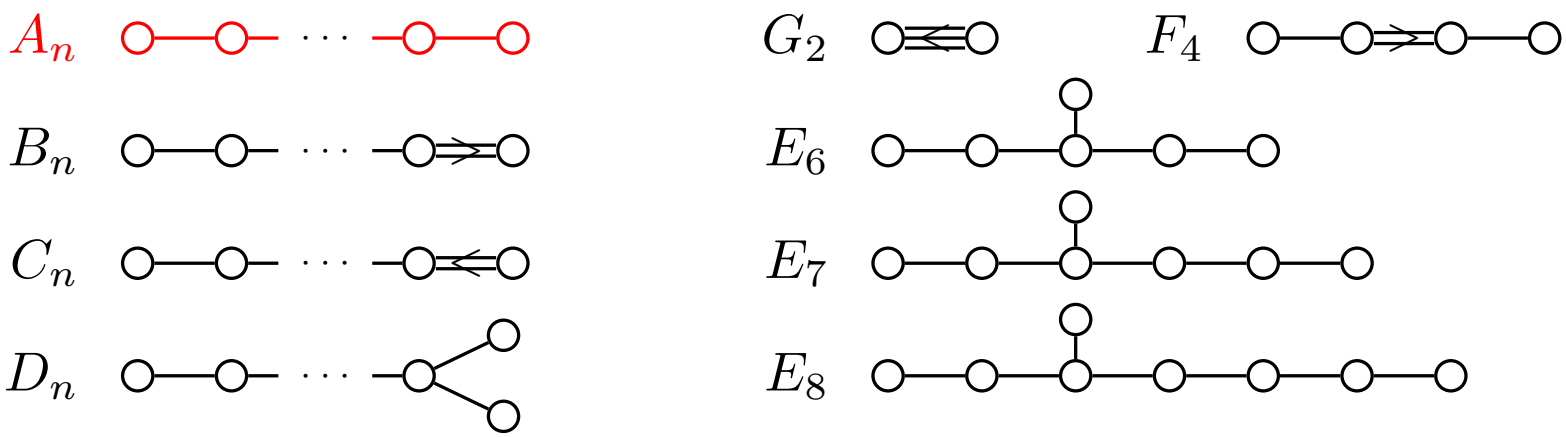
$$\epsilon_{\mu,j} = \sum_{a=1}^{r}\sum_{b=1}^{r}\sum_{\sigma\in S_r} \operatorname{sgn}(\sigma)\cdot$$

$$\binom{rr'+1}{r'+(\sigma(1)-1),\ldots,r'+(\sigma(a)-a)-j+1,\ldots,r'+(\sigma(b)-b)+j,\ldots,r'+(\sigma(r)-r)}$$

where $r' = n' - r$.

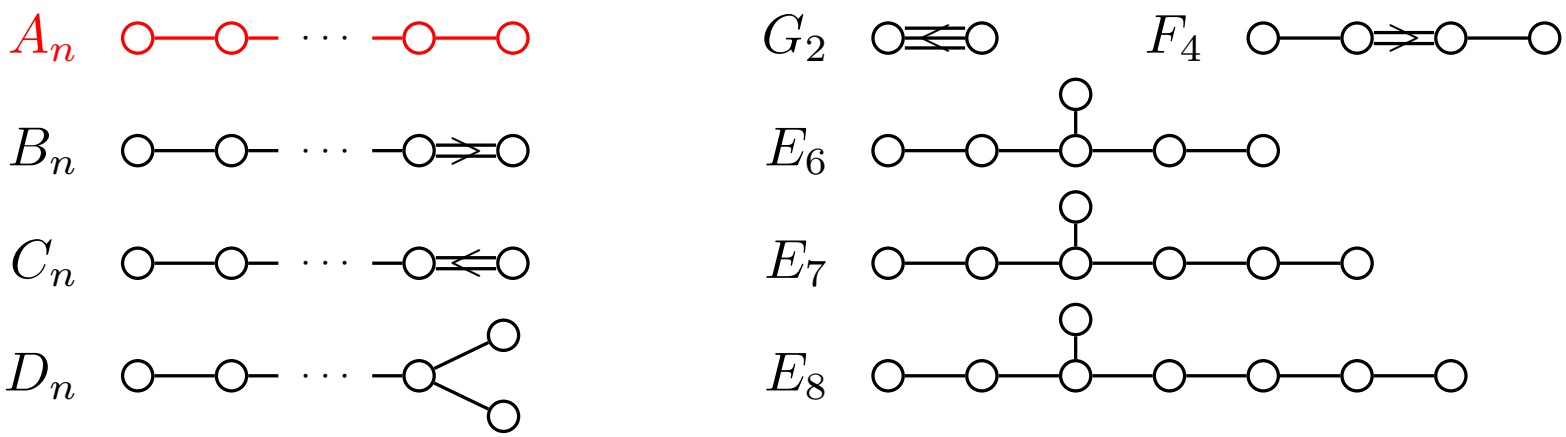
Dynkin diagrams

Every simple Lie algebra can be encoded by its Dynkin diagram:



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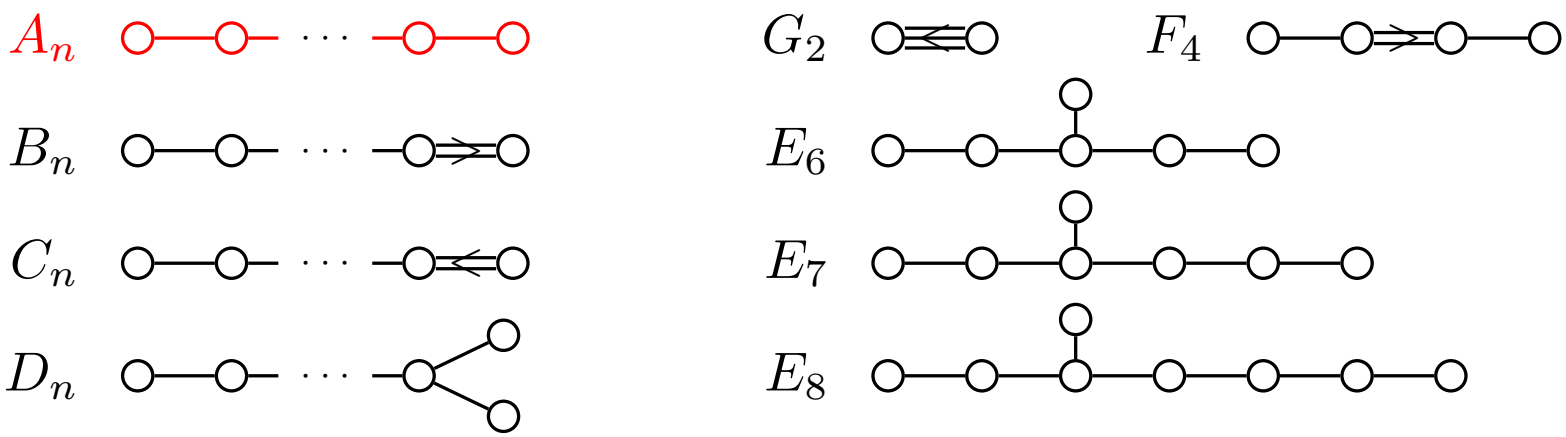


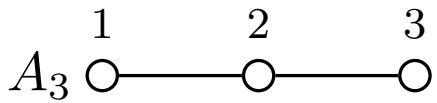
Definition (Simply laced)

The Cartan matrix is an $n \times n$ matrix with a 2 for every vertex (the diagonal) and -1's for edges.

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$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix} \end{array}$$

New approach

We can store the number of paths at vertex r on the Dynkin diagram in one function. $M_r = r(n' - r)$.

$$\tau_r(t) = \langle \exp(te)u_-, u_+ \rangle = (\#\text{paths}) \cdot \frac{t^{M_r}}{M_r!}$$

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Theorem (The Toda equation, Kostant, 1979)

$$\tau_r \tau_r'' - (\tau_r')^2 + \tau_{r-1}\tau_{r+1} = 0$$

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Let's check $\mathfrak{sl}_4$. We know already that $\tau_1 = \frac{t^3}{3!} = \tau_3$ and $\tau_2 = 2\frac{t^4}{4!}$.

$$\left(2\frac{t^4}{4!}\right)\left(2\frac{t^2}{2!}\right) - \left(2\frac{t^3}{3!}\right)^2 + \left(\frac{t^3}{3!}\right)^2 = \left(\frac{1}{12} - \frac{1}{9} + \frac{1}{36}\right) t^6 = 0$$

Eigenvectors

Second-order perturbation ($\xi^2 = 0 = \eta^2$):

$$\begin{aligned}\tau_r(t;\xi,\eta) &= \langle (1+\eta e_j)\exp\left(t(e+\xi f_j)\right) u_-,\ u_+\rangle \\ &= \tau_r(t)\cdot\Big(1 + \underbrace{\xi\, x_r\, t^{\,j+1}}_{\text{one } f_j} + \underbrace{\eta\, x_r'\, t^{-j}}_{\text{one } e_j} + \underbrace{\xi\eta\, K_r\, t}_{\text{both, } c\kappa_\mu(e_j,f_j)}\Big)\end{aligned}$$

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- This still satisfies the Toda equation!

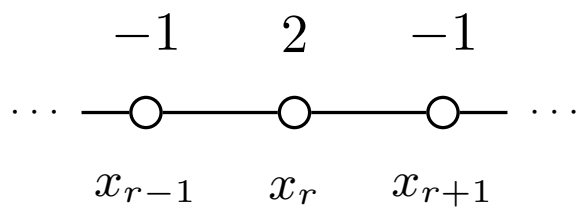
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- The ξ terms, with $D = \operatorname{diag}(M_r)$:

$$M_r\,\underbrace{(2x_r - x_{r-1} - x_{r+1})}_{\text{row } r \text{ of the Cartan matrix } A} = j(j+1)\,x_r \quad\Longleftrightarrow\quad DAx = j(j+1)\,x$$



Closed formula in type A_n

Theorem (Aricioglu–Wang, 2026)

Let $V_\mu = \bigwedge^r \mathbb{C}^{n'}$ and $M_r = r(n'-r)$. For $1 \le j \le n$,

$$\epsilon_{\mu,j} = \frac{(n-j)!\,(M_r+1)!}{(n+j+1)!} \prod_{i=0}^{r-1} \frac{i!}{(n'-r+i)!} \cdot K_r, \qquad \text{where}$$

$$K_r = M_r^2\, w(r)^2 + j^2(j+1)^2 \sum_{i=1}^{n} \frac{\min(r,i)\,(n'-\max(r,i))}{n'}\, w(i)^2,$$

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- **Steps**

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- Steps, **eigenvector**

Closed formula in type A_n

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Let $V_\mu = \bigwedge^r \mathbb{C}^{n'}$ and $M_r = r(n'-r)$. For $1 \le j \le n$,

$$\epsilon_{\mu,j} = \frac{(n-j)!\,(M_r+1)!}{(n+j+1)!} \prod_{i=0}^{r-1} \frac{i!}{(n'-r+i)!} \cdot K_r, \qquad \textit{where}$$

$$K_r = M_r^2\, w(r)^2 + j^2(j+1)^2 \sum_{i=1}^{n} \frac{\min(r,i)\,(n' - \max(r,i))}{n'}\, w(i)^2,$$

$$w(r) = \sum_{i=0}^{j-1} (-1)^i \frac{(n-1-i)!\,(i+j+1)!}{(n-j)!\,(i+1)!\,(j-1-i)!\,j(j+1)} \binom{r-1}{i}.$$

- Steps, eigenvector, $(\textbf{Cartan matrix})^{-1}_{r,i}$

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- All parts of this come from the Dynkin diagram!

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- **Steps**, **eigenvector**, **(Cartan matrix)$^{-1}_{r,i}$**
- All parts of this come from the Dynkin diagram!
- The strategy we just used for Type A_n works well for other types.

Acknowledgements

- I'm incredibly grateful for Zeyu Wang's mentorship and encouragement all throughout this year!
- I would also like to thank the PRIMES program for providing such an amazing opportunity.
- My mom and grandma for all their love and support ♡

Thank you!

Any questions?

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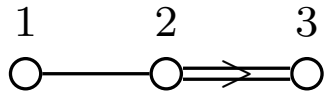
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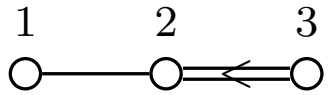
Extras

B_3



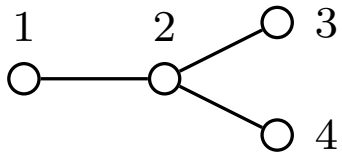
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C_3



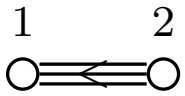
$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -2 & 2 \end{pmatrix} \end{array}$$

D_4



$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & -1 \\ 0 & -1 & 2 & 0 \\ 0 & -1 & 0 & 2 \end{pmatrix} \end{array}$$

G_2



$$\begin{array}{c} \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{cc} 1 & 2 \end{array} \\ \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix} \end{array}$$