Group Theory and Coxeter Groups

Lillian MacArthur, Joshua Park, Gina Li

MIT PRIMES Circle

May 2026

Introduction to Groups

Definition

A group is an ordered pair $(G, *)$ of a set G and a binary operation $*$ such that

1. there is an *identity* element $e \in G$ such that $a * e = e * a = a$ for all $a \in G$
2. for all $a \in G$, there is an *inverse* element a^{-1} such that $a * a^{-1} = a^{-1} * a = e$
3. the associative property holds, i.e., $a * (b * c) = (a * b) * c$ for all $a, b, c \in G$

Definition

Groups are not necessarily commutative, i.e., $a * b$ is not necessarily equal to $b * a$ for general $a, b \in G$. Groups that follow the commutative property are said to be *abelian*.

Definition

For a finite group G, we say that the *order of a group G* (written as $|G|$) is the number of elements in G.

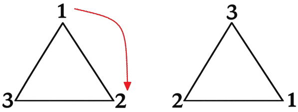
Definition

The *order of an element $a \in G$* (written as $|a|$)is the smallest $n \in \mathbb{N}$ such that $a^n = e$ (if such an n exists). In this case, a has *finite order*. If no such positive integer n exists, then a has *infinite order*. In particular, $|e| = 1$ because $e^1 = e$.

Dihedral Groups

Definition

A *symmetry* of the n-gon is a rigid motion that maps the n-gon onto itself.



Definition

The *composition* of two symmetries x and y, written $x \circ y$, is the symmetry obtained by performing y first and then x.

Definition

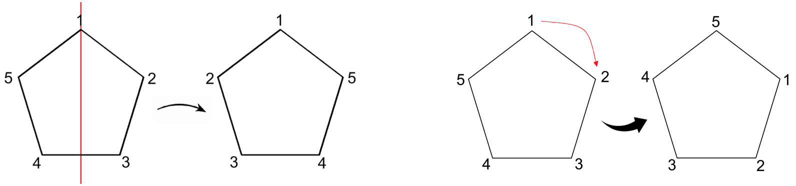
We denote the set of all symmetries of the regular n-gon by D_n.

Claim

D_n is a group (under composition)—we call this group the dihedral group. [2]

The Actions in D_n

Two simple examples of symmetry are the reflection (s) and the rotation (r) by $360/n$ degrees ($s, r \in D_n$).
For $n = 5$, these actions look like this:



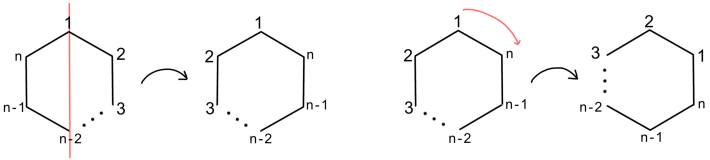
Claim

The rotation r has order n, and the reflection s order 2.

Continued

Claim

The dihedral group has order $2n$. Moreover, every element of D_n can be written as compositions of r and s.



Any element in D_n can be written as $r^k s$ or r^k.

Generators and Relations

Definition

A subset S of a group G is called a *generating set* for G if every element of G can be written as a finite product of elements of S and their inverses. The elements of S are called *generators* of G.

Definition

The *relations* in a group G are all of the equations that its generators satisfy.

Generators of the Dihedral Group

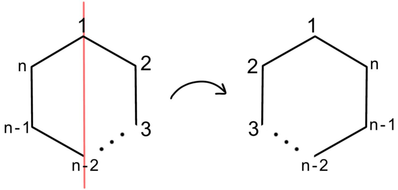
Because every symmetry of the n-gon in the dihedral group can be written in terms of r, s, these are generators. In fact, D_n is given by the following generators and relations:

Claim

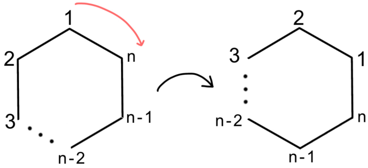
$D_n = \langle s, r \mid s^2 = e, r^n = e, rs = sr^{-1} \rangle$. *[2]*

$rs = sr^{-1}$

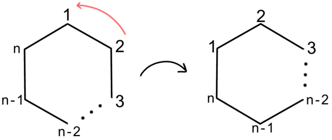
Flip (s)



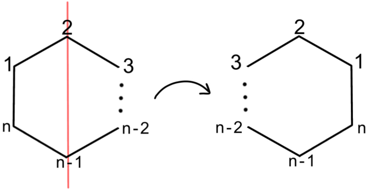
Clockwise rotation (r)



Counterclockwise rotation (r^{-1})



Flip (s)



Presentation of Groups

Definition

A *group presentation* is an expression $G = \langle S \mid R \rangle$, where S is a set of generators and R is a set of relations, such that G is the group given by these. [2]

Theorem (Adian-Rabin Theorem)

There does not exist an algorithm that takes in a group presentation and determines whether or not the group defined by this presentation is finite. [4]

Coxeter Groups

Definition

Let W be a group generated by set S. For generators $s_i, s_j \in S$, let m_{ij} denote the order of the product $s_i s_j$. W is a *Coxeter group* if it has the presentation

$$\langle s_1, s_2, ..., s_n \in S \mid (s_i s_j)^{m_{ij}} = e \rangle,$$

where $m_{ii} = 1$, $m_{ij} = m_{ji}$, and $m_{ij} > 1$ for $i \neq j$. [3]

Claim

The dihedral group is a Coxeter Group.

Proof.

$D_n = \langle s, r \mid s^2 = e, r^n = e, rs = sr^{-1} \rangle$ can also be presented as

$$\langle (s), (sr) \mid (s)^2 = e,\ (sr)^2 = e,\ (s \cdot sr)^n = e \rangle.$$

□

Continuation

Definition

A *Coxeter matrix* is an $n \times n$ matrix $M = (m_{ij})_{1 \leq i,j \leq n}$ with natural number entries such that $m_{ii} = 1$ and $m_{ij} = m_{ji}$ is greater than 1 for all $i \neq j$. Thus, we can view each entry as the order of a product of generators of the Coxeter group. [3]

The Coxeter group together with its generating set, (W, S), is called the Coxeter system of type M.

Example D_n is a Coxeter system of type $\begin{bmatrix} 1 & n \\ n & 1 \end{bmatrix}$.

Example The Coxeter system of type $\begin{bmatrix} 1 & 3 & 2 \\ 3 & 1 & 4 \\ 2 & 4 & 1 \end{bmatrix}$ is

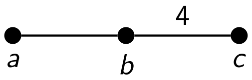
$$\langle a, b, c \mid a^2 = b^2 = c^2 = (ab)^3 = (ac)^2 = (bc)^4 = e \rangle.$$

Coxeter Diagrams

To construct a diagram from a Coxeter system (W, S) of type M, we follow the rules below:

1. If $m_{ij} = 2$, no edge is drawn
2. If $m_{ij} = 3$, an unlabeled edge is drawn
3. If $m_{ij} > 3$, an edge is drawn and labeled with the value of m_{ij} [3]

Example. The Coxeter diagram for $\langle a, b, c | a^2 = b^2 = c^2 = (ab)^3 = (ac)^2 = (bc)^4 = e \rangle$ is



Finite Coxeter Groups

Claim

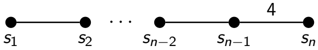
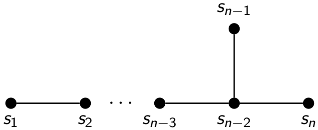
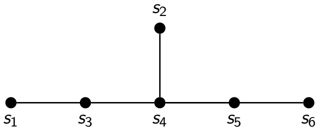
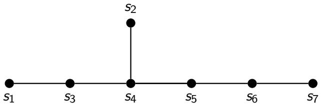
There are only finitely many finite Coxeter Groups (that is, Coxeter Groups of finite order).

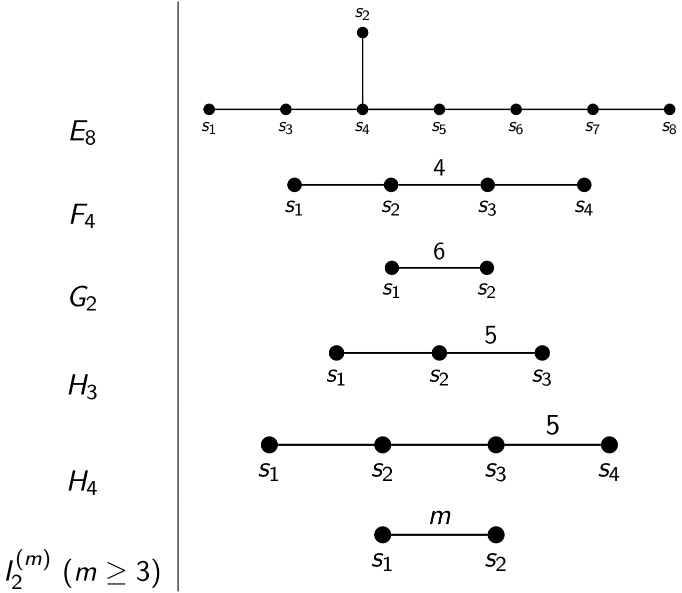
Recall that Adian-Rabin theorem that there cannot exist a general algorithm for determining whether a given Group Presentation presents a finite group. This result is given by realizing the Coxeter Groups as symmetry groups, where the generators act by reflections. If you want to learn more, read our paper on the MIT PRIMES Circle website!

Claim

The only finite Coxeter groups are the groups of type $A, B, C, D, E, F, G, H, I$*, whose diagrams we will be showing in the following slide. [1]*

All Finite Coxeter Groups

Name	Diagram
A_n $(n \geq 1)$	
$B_n = C_n$ $(n \geq 3)$	
D_n $(n \geq 4)$	
E_6	
E_7	



E_8
s_2
s_1 s_3 s_4 s_5 s_6 s_7 s_8
F_4
4
s_1 s_2 s_3 s_4
G_2
6
s_1 s_2
H_3
5
s_1 s_2 s_3
H_4
5
s_1 s_2 s_3 s_4
$I_2^{(m)}$ $(m \geq 3)$
m
s_1 s_2

We would like to thank Lilian MacArthur for mentoring us this semester and the PRIMES Circle coordinators, Paige and Mary, for providing us this opportunity.

Thank you!
Any questions?

References

[1] Anders Bjorner and Francesco Brenti. *Combinatorics of Coxeter Groups*. Springer Press, 2005.

[2] David S. Dummit and Richard M. Foote. *Abstract Algebra*. 3rd ed. New York: Wiley, 2004.

[3] Shourya Mukherjee. *Classification of Finite Coxeter Groups*. URL: https://math.uchicago.edu/~may/REU2019/REUPapers/Mukherjee.pdf. (accessed: 04.26.2026).

[4] Carl-Fredrik Nyberg-Brodda. *The Adian-Rabin Theorem – An English translation*. 2024. arXiv: 2208.08560 [math.GR]. URL: https://arxiv.org/abs/2208.08560.