Introduction to Graph Theory: Spanning Trees

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Introduction

Overview

Graphs are versatile tools for analyzing a world full of networks. Any systems that involve "nodes", such as cities, radio towers, etc. and their connections are improved by properties derived from graph theory.

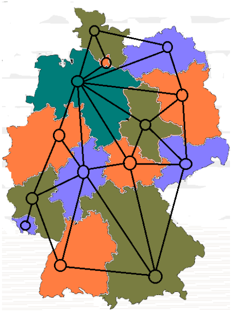

Figure 1: Map of the provinces of Germany overlaid with the respective graph.

Key Definitions

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A **subgraph** of a graph G is a graph G' whose vertex set and edge set are subsets of the vertex and edge sets, respectively, of G.

More Key Definitions

Definition

Cycles are a sequence of vertices connected by edges, where it is possible to return to the first without repeating edges.

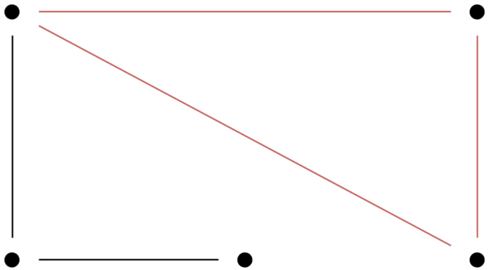

Figure 2: A graph containing a cycle in red.

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Connected graphs are graphs where there is a path between any two vertices.

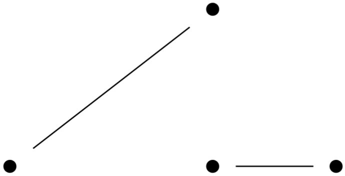

Figure 3: A disconnected graph

Introduction to Trees

Definition

Trees are connected graphs with no cycles.

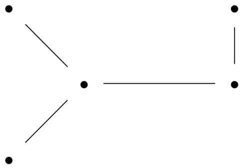

Figure 4: A Tree

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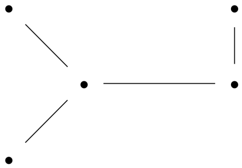


Figure 4: A Tree

Theorem

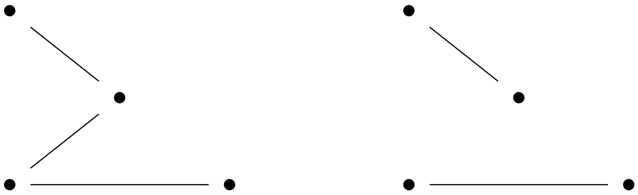
A tree with n vertices must have $n - 1$ edges.

Inductive Proof: Base Case

We can see that all trees have $n-1$ edges when $n=1$ or$n=2$.

Inductive Step

We can now consider a tree T on $n+1$ vertices, given that the property holds for trees with n vertices or fewer.



Induction Process on More Vertices

Removing an edge from a tree always creates two trees with fewer vertices. Since each tree has fewer vertices, the property will hold on both of them. This is an example for when $n = 4$.

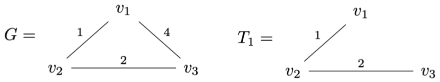
Algebraic Step

$m(T) = m(T_1) + m(T_2) + 1 = (n(T_1) - 1) + (n(T_2) - 1) + 1 = n(T) - 1$

Spanning Trees

Definition

A **spanning tree** T of a graph G is a subgraph of G where the entirety of the set of vertices is preserved, and T is a tree.



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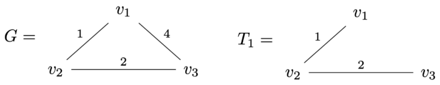

Definition

Weighted graphs are graphs in which each edge has an assigned value called its weight. If an edge, denoted by e, belongs to a graph, we denote its weight by $w(e)$. Weight allows an edge to hold information about itself, which is particularly useful in optimization problems such as pathing.

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Definition

Minimum spanning tree is a spanning tree T of a graph G such that the sum of the weights of T are minimized. There may be multiple minimum spanning trees of a graph.

Spanning Tree Diagrams

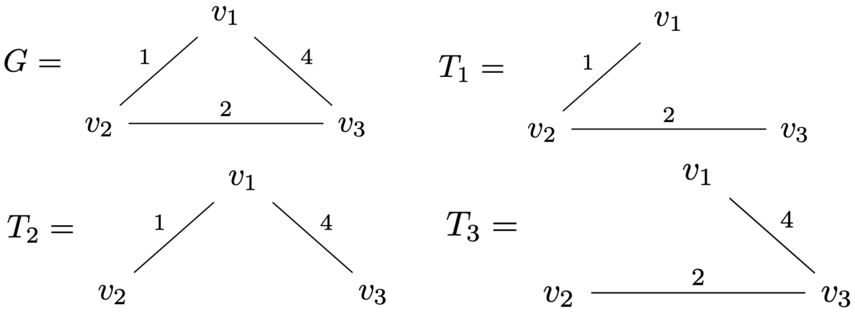
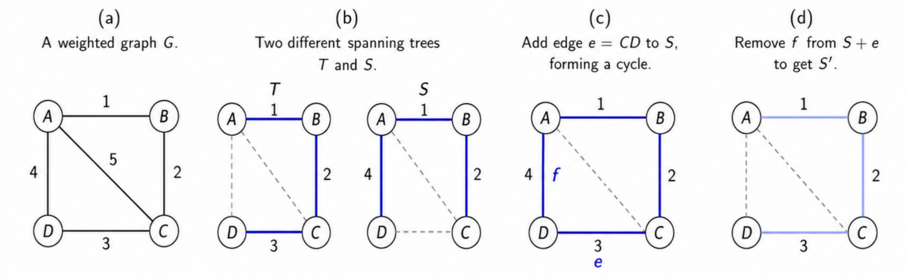


Figure 5: Different spanning trees of G, T_1, T_2, T_3. The sum of the weights of a spanning tree is simply the sum of the edge weights, $w(T) = \sum_{e \in E(T)} w(e)$. So if we find the weight of each tree then we find that T_1 is the minimum spanning tree with a weight of 3.

Uniqueness of Minimum Spanning Trees

Theorem

If all edge weights are distinct, there is a unique minimum spanning tree.



Proof.

Suppose G has two different MSTs, T and S, and let $e \in T$, $e \notin S$ be the lightest edge where they differ. Adding e to S creates a cycle with some edge $f \in S$, $f \notin T$, so $w(e) < w(f)$ and replacing gives $S' = S \cup \{e\} - \{f\}$ with $w(S') = w(S) + w(e) - w(f) < w(S)$, a contradiction. □

Prim's Algorithm

Definition

Prim's Algorithm is a greedy algorithm used to find the minimum spanning tree of a graph.
Let G be a connected weighted graph. Choose a starting vertex. Let S be the set of vertices already included in the growing tree. Among all edges with one endpoint in S and one endpoint in $V(G) \setminus S$, choose an edge of minimum weight. Add this edge and its new endpoint to the tree. Repeat until all vertices have been included.

Prim's Algorithm: Continued

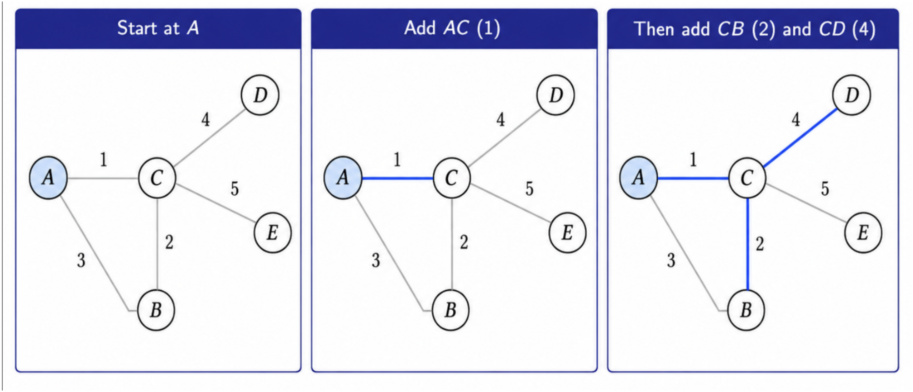


Figure 6: First selections of Prim's Algorithm

Prim's Algorithm: Continued

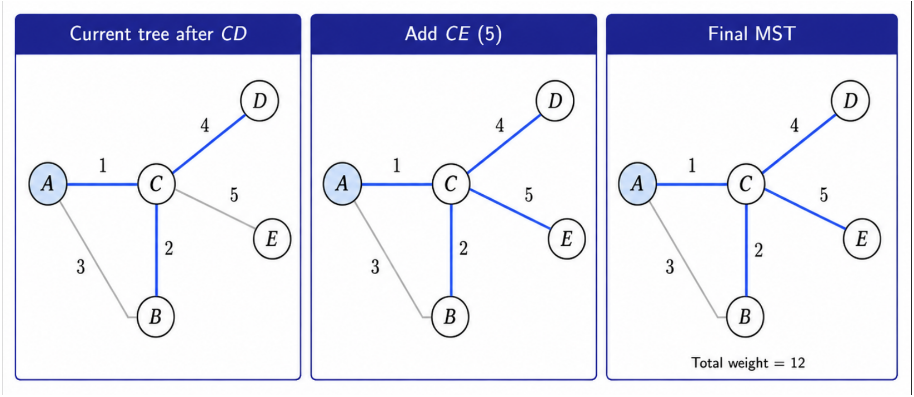


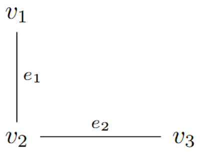
Figure 7: Final selections of Prim's Algorithm

Spectral Theory

Definition

The **adjacency matrix** A is an $n \times n$ matrix such that every entry a_{ij} is equal to the number of edges between v_i and v_j. As we can see here, the adjacency matrix for the below graph is:

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

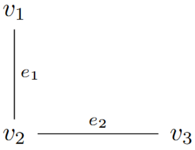


More Spectral Theory

Definition

The **Incidence matrix**, B, is the $n \times m$ matrix where the rows represent the vertices and columns represent edges. Each edge (column) has two non-0 values: a 1 and a -1, at the vertices it touches. For example, for the below graph, the incidence matrix is:

$$B = \begin{bmatrix} 1 & 0 \\ -1 & 1 \\ 0 & -1 \end{bmatrix}$$

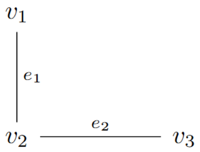


Laplacian

Definition

The **Laplacian**, L, is an $n \times n$ matrix such that $L =$diag$(d_1, ..., d_n) - A$. The **reduced Laplacian**, shown by L', is an $(n-1) \times (n-1)$ matrix obtained by removing the ith row and column of the matrix, $1 \leq i \leq n$.

$$L = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$



Incidence and Laplacian

Theorem

$L = B \cdot B^T$, $L' = B' \cdot B'^T$

Proof.

Every a_{ij} entry in $B \cdot B^T$ is the negative number of edges shared by v_i and v_j. For $v_i \neq v_j$, that is just the number of edges between them, and for $v_i = v_j$, that is the degree. That gives $B \cdot B^T =$diag$(d_1, ..., d_n) - A$. □

Necessary Theorems

Theorem

Cauchy–Binet. Where $A = m \times n$, $B = n \times m$ *matrices,* $A_{[m],S}$ *is a set of* m *columns and* $B_{S,[m]}$ *is a set of* m *rows:*

$$\det(AB) = \sum_{S \subset \binom{[n]}{m}} \det(A_{[m],S}) \det(B_{S,[m]})$$

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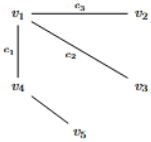
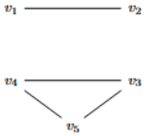
Proof.

If there are sections, the rows with the sections will add up to 0, so the determinant will be 0. If there is only 1 section, then there will be at least one row with only a 1 (or -1) entry, which will row reduce into one 0 but for B', there won't be a 0, leading to a determinant of ± 1. □

Example

$$B = \begin{bmatrix} 1 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \mathit{and} \left[\begin{array}{cccc} 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ \hline 0 & 1 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & -1 \end{array}\right]$$





Kirchoff's Matrix Tree Theorem

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Proof.

per Cauchy-Binet, we can say

$$\det(L') = \sum_{\substack{S \subset [m] \\ |S| = n-1}} \det(B'_S) \det(B'^T_S) = \sum_{\substack{S \subset [m] \\ |S| = n-1}} \det(B'_S)^2$$

Then, as $\det(B'_S) = \pm 1$ if and only if it is a spanning tree, $\det(B'_S)^2$ effectively adds 1 for every spanning tree, so it is equal to the number of spanning trees. □

References

B. Balakrishnan and K. Ranganathan. A Textbook of Graph Theory. Springer, 2012

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