DYNAMICAL SYSTEMS

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Dynamical Systems

A **dynamical system** describes how different starting conditions evolve over time.

One example is interest, using the function $F(x) = 1.1x$.

If we put x dollars in the bank, we will have $F^n(x) = 1.1^n x$ dollars after n years.

The Quadratic Map

Another example is the quadratic function, $Q_c(x) = x^2 + c$.

This function will have different behavior for different values of the parameter, c.

Orbits

When we iterate a function on a point x_0, we get a sequence of points, called the **orbit** of x_0 under the function.

The orbit of x_0 under F is the sequence $x_0, F(x_0), F^2(x_0), \ldots$

For example, the orbit of 100 under our interest function $F(x) = 1.1x$ is $100, 110, 121, 133.1 \ldots$

Fixed Points

A **fixed point** of a function is a point for which $F(x) = x$.

The orbit of a fixed point x_0 looks like $x_0, x_0, x_0, \ldots$

For example, 0 is a fixed point of our interest function.

Fixed Points of Q_c

To find the fixed points of Q_c, we solve the equation $Q_c(x) = x$. We have:

$$\begin{aligned} Q_c(x) &= x \\ x^2 + c &= x \\ x^2 - x + c &= 0 \\ x &= p_\pm = \frac{1 \pm \sqrt{1-4c}}{2} \end{aligned}$$

Note that these fixed points only exist for $c \le \frac{1}{4}$.

Periodic Points

A **periodic point** (or n-**cycle**) of a function F for which $F^n(x) = x$ for some n.

We call n the **period** of such a point.

The orbit of an n-cycle looks like $x_0, x_1, x_2, \ldots, x_n, x_0, x_1, \ldots$

The 2-Cycle of Q_c

To see if Q_c has a 2-cycle, we solve the equation $Q_c^2(x) = x$:

$$Q_c^2(x) = (x^2 + c)^2 = x$$
$$x^4 + 2cx^2 - x + c^2 + c = 0$$

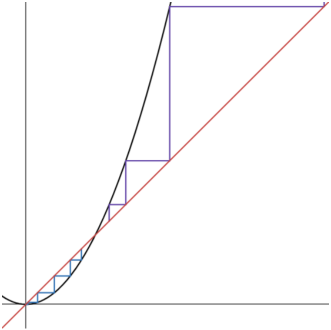
Two of the roots of this polynomial are the fixed points we already found; the other two roots are the 2-cycle

$$q_{\pm} = \frac{-1 \pm \sqrt{-4c - 3}}{2}.$$

This cycle only exists for $c < -\frac{3}{4}$.

Attraction and Repulsion

A fixed point x of a function F is **attracting** if $|F'(x)| < 1$, **repelling** if $|F'(x)| > 1$, and **neutral** if $|F'(x)| = 1$.



The definitions are the same for n-cycles, but replace F with F^n.

Attracting and Repelling Fixed Points of Q_c

To determine whether Q_c's fixed points are attracting or repelling, we evaluate their derivatives. We have $Q_c'(x) = 2x$, so

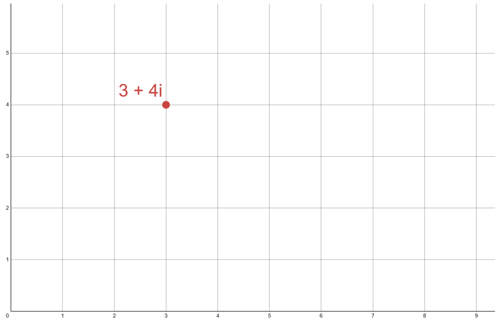
$$Q_c(p_{\pm} = Q_c'\left(\frac{1 \pm \sqrt{1-4c}}{2}\right) = 1 \pm \sqrt{1-4c}.$$

From this, we see that:

- the larger fixed point is neutral for $c = \frac{1}{4}$ and repelling for $c > \frac{1}{4}$;
- and the smaller fixed point is neutral for $c = \frac{1}{4}$, attracting for $-\frac{5}{4} < c < \frac{1}{4}$, neutral at $c = -\frac{5}{4}$, and repelling for $c < -\frac{5}{4}$.

The Complex Plane

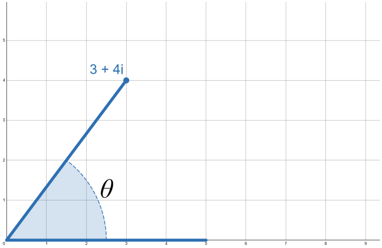
We define a **complex number** to be any number $a + bi$ where $a, b \in \mathbb{R}$ and $i = \sqrt{-1}$. We can plot a complex number $a + bi$ on the coordinate plane at (a, b)



Modulus, Argument, and $e^{i\theta}$

The **modulus** of a complex number z, $|z|$, is defined to be a complex number's distance from the origin. The **argument** of a complex number is the angle the number makes with the x-axis.

The argument and modulus can represent any complex number in a form $re^{i\theta}$, relying on Euler's formula $e^{i\theta} = \cos\theta + i\sin\theta$.

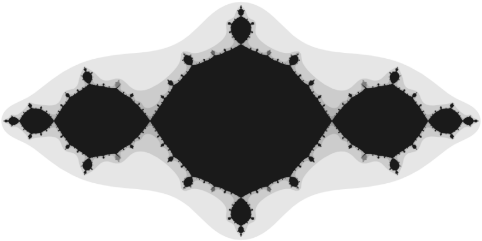


Julia Sets

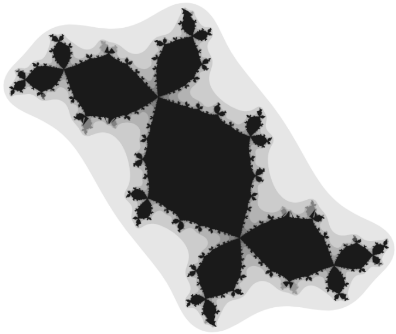
We can analyze Q_c as a complex-valued function too, letting c take any value in $\mathbb{C}$.
A **Julia set** for a specific c value is the set of all complex input values for z in which $Q_c(z)$ stays bounded.

Julia sets are fractals, meaning they are shapes with infinite complexity and possibly self-similarity.

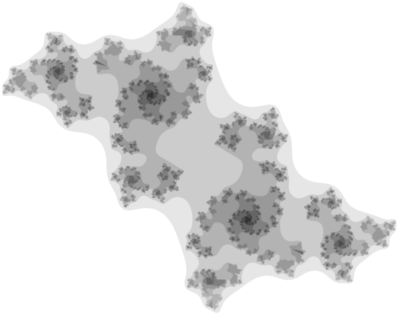
Julia Set: $c = -1 + 0i$



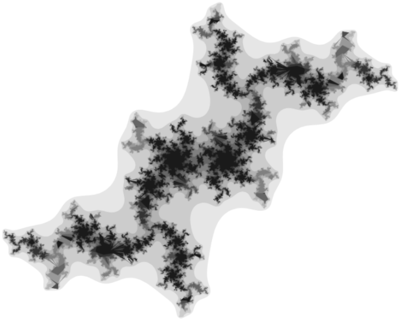
Julia Set: $c = -0.122561 + 0.744862i$



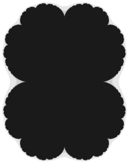
Julia Set: $c = -0.257 + 0.7i$

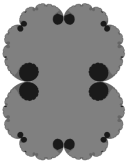


Julia Set: $c = -0.22 - 0.78i$



The Fundamental Dichotomy: 0.25 vs. 0.251





The Fundamental Dichotomy

The **Fundamental Dichotomy** states the following:

1. If the orbit of 0 under Q_c escapes to infinity then the Julia set of c contains infinitely many disjoint components.
2. If the orbit of 0 under Q_c stays bounded then the Julia set of c is one connected piece.

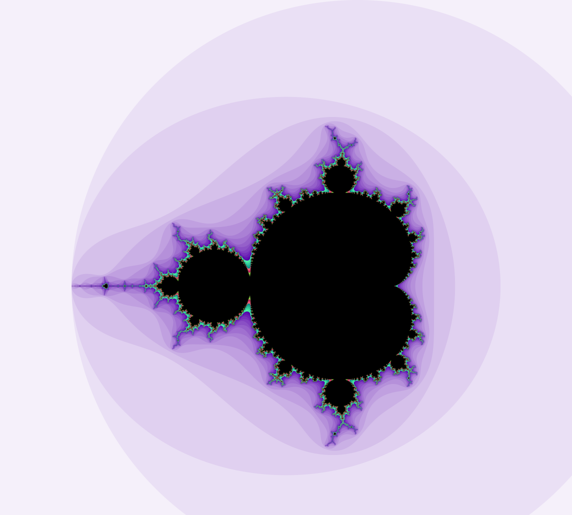
It is important to note this means there will never be a finite number of pieces other than 1 in a Julia set.

The Mandelbrot Set

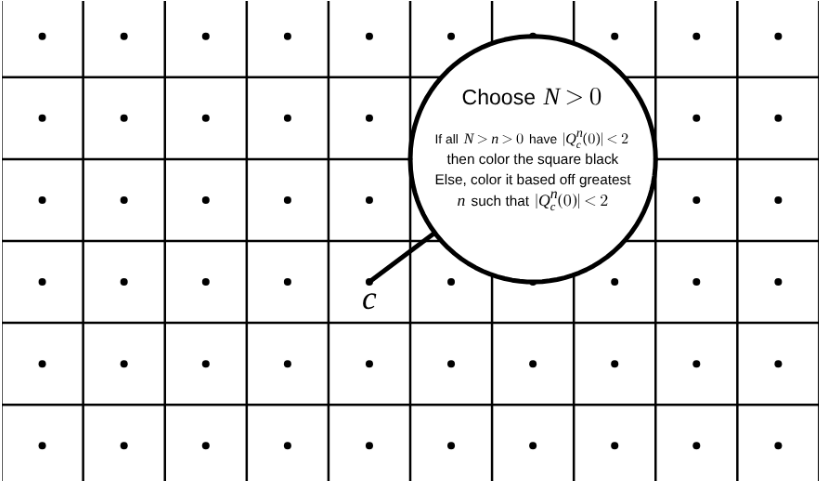
The **Mandelbrot set** is defined as the set of all c-values for which the respective Julia set is connected. Like the Julia sets, the Mandelbrot set is a fractal.

Unlike the Julia sets, which are in the input plane, the Mandelbrot set is in the parameter plane, meaning points represent values of c rather than z.

The Mandelbrot Set



Computation of the Mandelbrot Set



The Main Cardioid

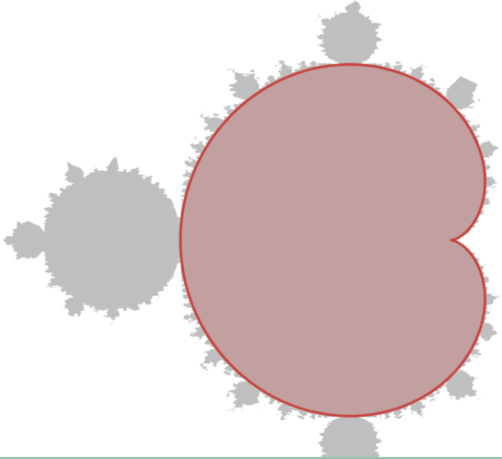
The main cardioid is the region of the Mandelbrot set for c-values where Q_c has an attracting fixed point.

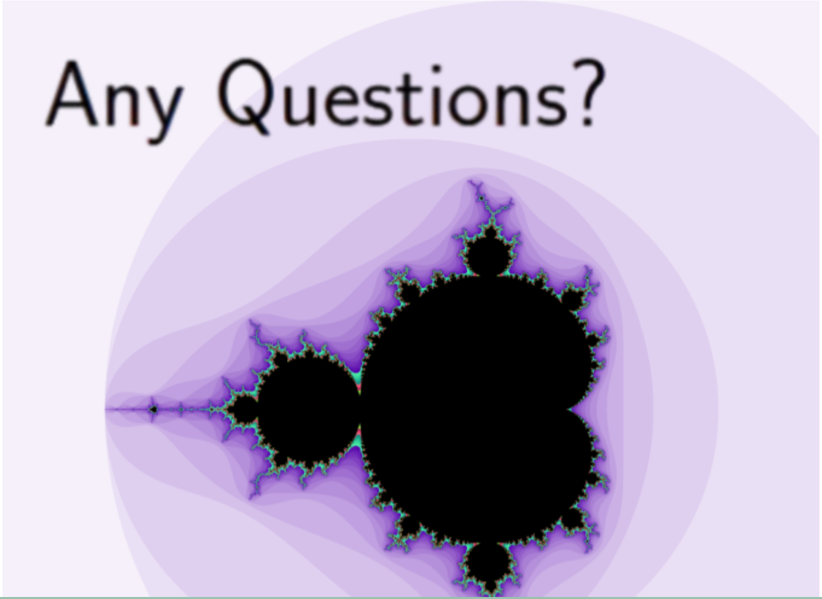
Any such point z must satisfy $z^2 + c = z$ and $|Q_c'(z)| = |2z| < 1$. The boundary of the inequality is given by $z = \frac{1}{2}e^{i\theta}$ for $\theta \in [0, 2\pi)$.

Since $c = z - z^2$ by the first equation, the boundary of this region is given by $c = \zeta(\theta) = \dfrac{1}{2}e^{2\pi i\theta} - \dfrac{1}{4}e^{4\pi i\theta}$.

The Main Cardioid

Here is the cardioid, visually:




Any Questions?

Sources

[1] *A First Course in Chaotic Dynamical Systems* by Robert L. Devaney