Graph Theory: Trees and Colorings

PRIMES Circle Program

Chloe Carrano and Eileen Lee

May 17, 2026

Introduction to Graph Theory

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- You and your friends are named: Alice, Bonnie, Chloe, Dan, and Eileen.
- Let's try to find a seating arrangement where everyone sits next to someone they know.

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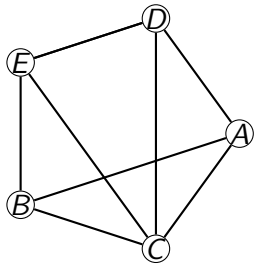

Figure: Graph representation

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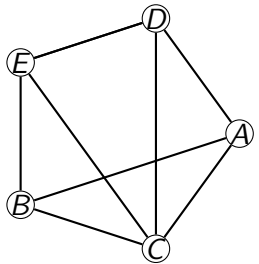

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- The graph representation offers us a way to see the possible seatings.

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Definition

The **degree** of a vertex v, denoted $\deg(v)$, is the number of edges incident with v.

Tree Graphs

with

Applications to Computer Science

Chloe Carrano

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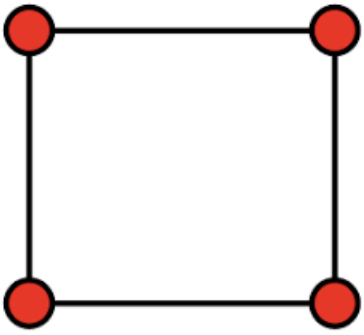
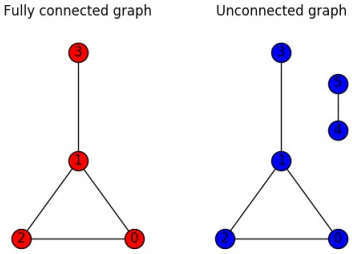
A graph is described to be **acyclic** if it contains no cycles.

Cycles and Connected Graphs



(a) A cycle with 4 vertices (4-cycle)

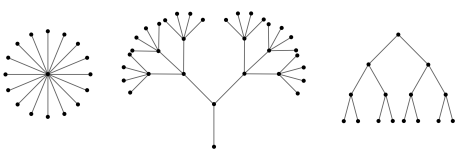


(b) Two graphs showing the definition of connectedness

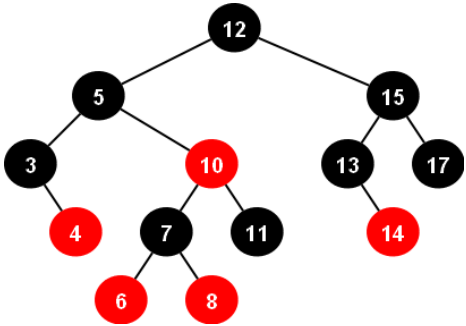
We represent a cycle graph with n vertices ($n \geq 3$) connected in a single closed chain with the notation C_n.

Trees

Again, tree graphs must be connected and contain no cycles! They may be colored, weighted, or sorted by vertex value (more on that later...)

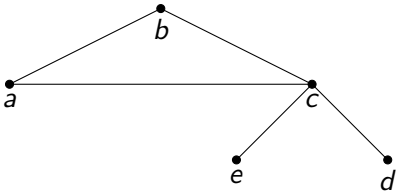


(a) Examples of trees – we would call these (as an entire group) a forest.



(b) A particular type of tree (red-black tree) with colored, weighted vertices.

Spanning Trees



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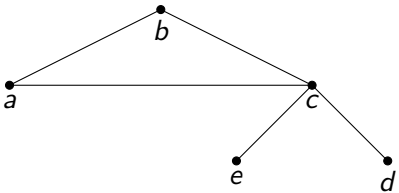

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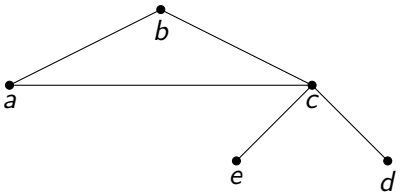

- Imagine you're in charge of designing a new network of train lines for the MBTA connecting five cities: Agria, Biloti, Clodo, Durley, and Emmio.
- One day, a tree falls on the train tracks between Biloti and Clodo. It looks like we can still get from any one city to another, even with this broken track! However, with any second, additional track broken, that wouldn't be possible.

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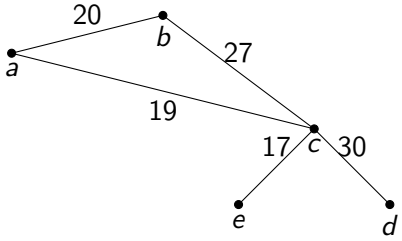
Definition

A spanning subgraph H of a connected graph G such that H is a tree is called a **spanning tree** of G.

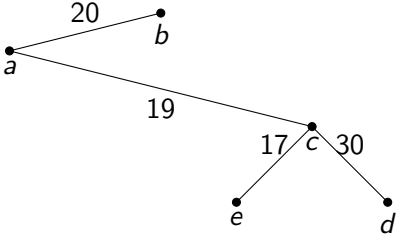
- If we want to minimize the cost of building entirely new train tracks, we can weight each edge on the graph with the cost it might take to rebuild:



Spanning Trees, Continued

- To minimize costs, we want to find the **minimum spanning tree** within this graph.
- We can use several different algorithms to find the minimum spanning tree of a weighted graph, but for now we'll use Kruskal's algorithm.

Using Kruskal's, we'll continuously select the lowest-weight available edge that connects two disconnected components (while ensuring no cycles are formed) until all vertices are connected. We're left with the following graph:



Kirchhoff's Matrix Tree Theorem

Gustav Kirchhoff, a German physicist and revolutionary electrical engineer, developed a theorem involving matrices to determine the total number of distinct spanning trees in a given graph – we call this Kirchhoff's Matrix Tree Theorem.

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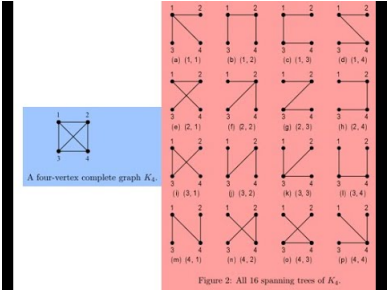

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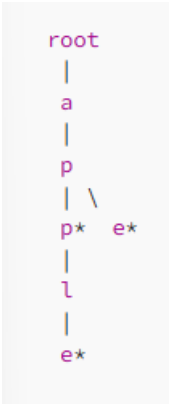

Figure: A prefix tree storing the strings "app," "ape," and "apple."

Applications of Trees

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Tree structures are everywhere – the next time that you see one in your daily life, think of the graph theory concepts behind their construction and properties!

Coloring Graphs

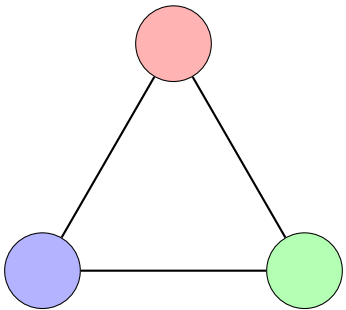
and

Ramsey Numbers

Eileen Lee

Part I

Graph Colorings



Vertex Coloring, Edge Coloring, and Applications

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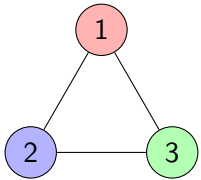

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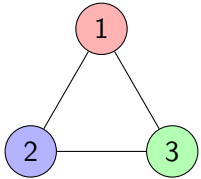

$$\chi(C_3) = 3$$

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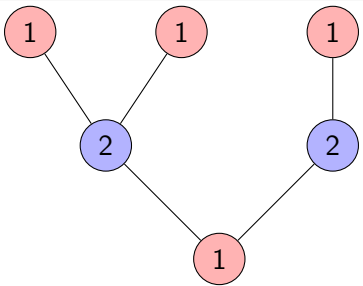

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This gives an upper bound on the chromatic number.

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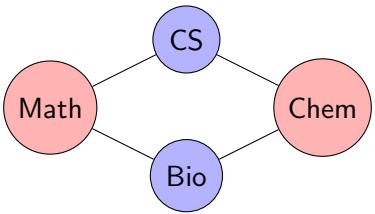

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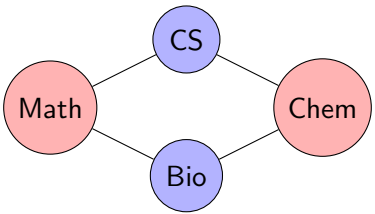

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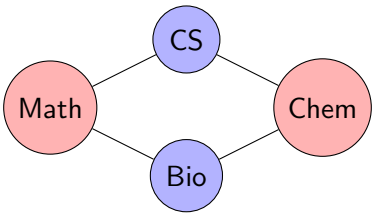

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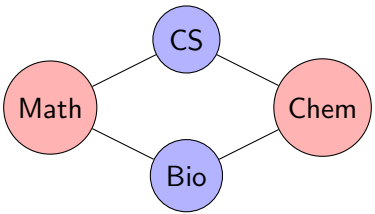

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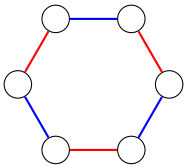

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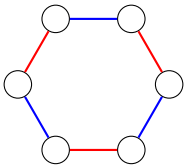

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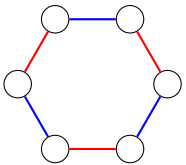

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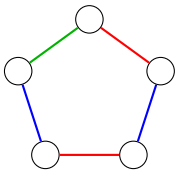
Odd Cycle



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Part II

Ramsey Numbers

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The Famous Result $r(K_3, K_3) = 6$

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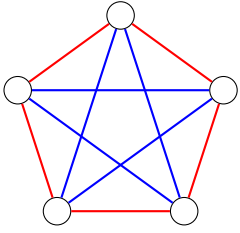
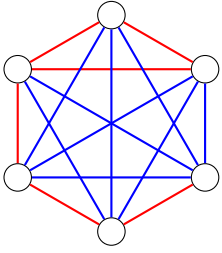
K_5



Avoids monochromatic triangles

K_6



A monochromatic triangle appears

Protein–Protein Interaction Networks

Protein–protein interaction networks model how proteins inside a cell communicate and work together.

- Vertices represent proteins

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These networks help biologists study how groups of proteins coordinate important cellular functions.

Proteins in the Model

We consider several proteins involved in cell regulation and signaling:

- *TP53*
- *MDM2*
- *RB1*
- *CDK2*
- *EGFR*
- *GRB2*
- *SOS1*

Interaction Types in the Network

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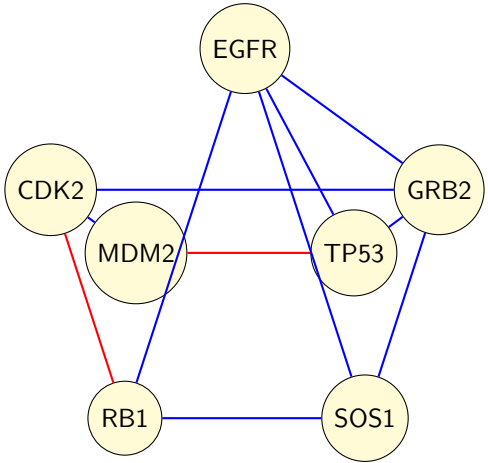

Figure: Red edges represent strong interactions, while blue edges represent weaker signaling interactions.

Ramsey Theory in Biological Networks

- Although the network appears irregular at first glance, Ramsey theory implies that complete disorder cannot persist in sufficiently large systems.

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- Eventually, subsets of proteins must appear in which interactions become uniform.

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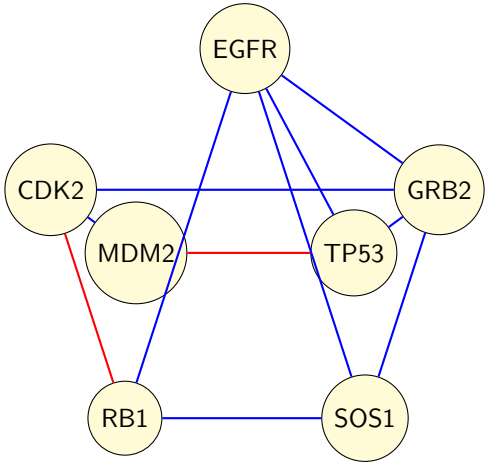

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  - stable protein complexes
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  - functional cellular modules

Ramsey Theory in Biological Networks

- These structures correspond biologically to:
  - stable protein complexes
  - signaling pathways
  - functional cellular modules
- Ramsey theory therefore provides a mathematical explanation for why biological systems naturally organize into clusters and coordinated pathways.

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- Graph coloring can be used for solving optimization problems.
- Ramsey theory shows patterns inevitably emerge.
- These ideas connect mathematics with:
  - computer science
  - biology
  - social sciences
  - etc

Questions?