# Self-Referential and Immensely Questionable Tests

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# Table of Contents

# 1

# Percentage Tests

# What are IQ Tests?

*If you choose the answer to this question at random, what is the probability you will be correct?*

a) 25%

b) 50%

c) 60%

d) 25%

None of these are correct! You had a 50% chance of getting 25%, 25% chance of getting 50% and a 25% chance of getting 60%

# Definitions

a) 25%

b) 50%

c) 60%

d) 25%

There are **4** answer choices (a, b, c, d)

There are **3** answers (25%, 50%, 60%)

Essentially, **answers** are distinct **answer choices**.

# Formalizing tests

Define $K$ to be the number of answer choices for a test. Define the options, in percentage points, as:

$$a_1, a_2, \ldots, a_K$$

We define the cost as $C = \sum_{i=1}^{K} a_i$

# An Example

As an example, take this the test from before:

**25%, 50%, 60%, 25%**

This test has a cost $C = 25 + 50 + 60 + 25 = 160$ and number of answer choices $K = 4$.

# Solvable Percentage Tests

- We call an answer *a* **valid** if $a = \frac{100m}{K}$, where *m* is the number of times that *a* appears and *K* is the total number of answers.
- We call a test **solvable** if it contains at least one valid answer.
- For example, the test with answers 30%, 50%, 50%, 75% is solvable, since there is a 50% chance of choosing the answer choice 50.

# *Flaws with percentage tests*

- Some options can imply the possibility of a non-integer number of valid answers
- Adding an extra option changes the whole test

**(A) 25% (B) 50% (C) 50% (D) 60%**

Selecting option D implies that there are 2.4 correct answers, clearly extraneous

# 2

# Natural Tests

# Natural tests

- Percentage tests asks what the probability the chosen random answer choice is correct
- Natural Tests now ask "How many correct answers are there?"

25, 50, 50, 75 ⇒ 1, 2, 2, 3

For this test, C=200 For this test, C=8

25, 50, 50, 60 ⇒ 1, 2, 2, **2.4**

# Solvable Natural Tests

- We call an answer *a* **valid** if $a = m$, where $m$ is the number of times $a$ appears in the natural test.
- We say a natural test is **solvable** if it consists of at least one valid answer
- An example:
    - Which answers are valid in the following test?
        - 1, 2, 2, 4, 4, 4, 5

# 3

# Immensely Questionable Tests

# Another Puzzle

**How many answer choices are correct?**

# 1, 2, 2, 3

# Yet Another Puzzle

**How many answer choices are correct?**

# 1, 2, 2, 4

# Para vs. Meta: Two kinds of self-reference

Paradoxical:

**1, 2, 2, 3**

1 is valid, 2 is valid. So there are 3 correct answers, so 3 is a correct answer now.

Meta:

**1, 2, 2, 4**

1 is valid and 2 is valid. If we were to include 4, then 4 of the answer choices would be correct, so 4 is correct.

# Formalizing Para Valid Answers

For a given test with *k* total valid answer choices, we define an answer *a* to be para valid if $a=k$.

# Formalizing Meta Valid Answers

For a given test with $k$ total valid answer choices, we define an answer $a$ with multiplicity $m$ to be meta valid if $a=m+k$

# States

- We define states as the set of answers we consider to be “correct”
- More rigorous way of solving complicated meta and para tests
- Each state is calculated based on the previous state

# *Para as an answer selection method*

# 1, 2, 2, 3, 4, 6

We consider [illegible] correct answers.

**Stage [illegible]**

# *Meta as an answer selection method*

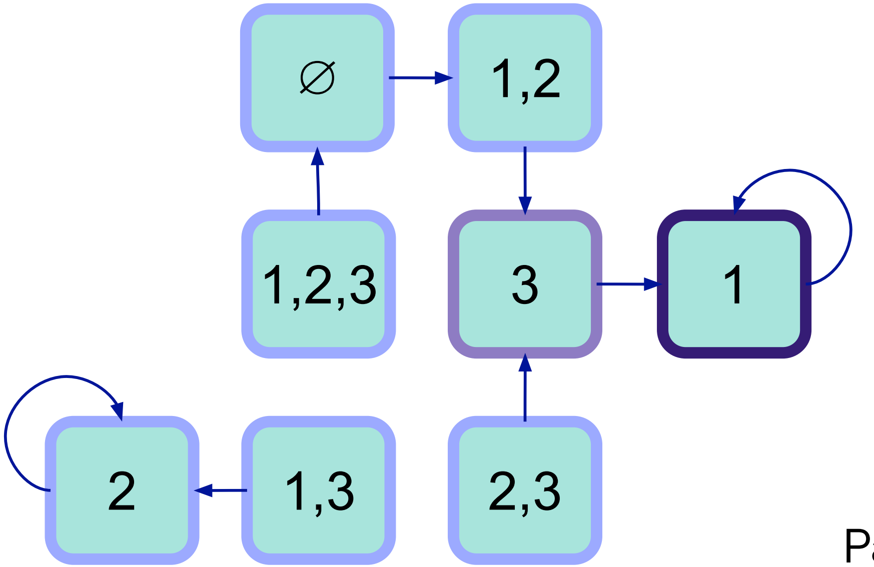
# 1, 3, 3, 3, 6, 6, 8, 8

We consider 1, [illegible] correct answers.

# Test Diagrams

- A visualization of a test
- The vertices are the sets of answers

# A Para-Test Diagram




**1, 2, 2, 3**

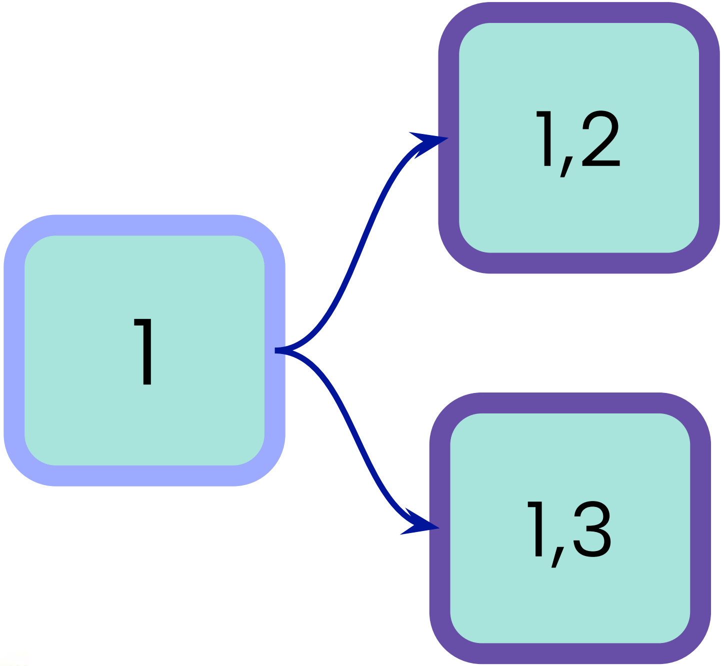
Para-graph
of the test
1, 2, 2, 3

# Branching

Within meta tests, there is a concept of branching. Let's consider the test:

**1, 2, 3, 3**

# A Branching Test Diagram




# 1, 2, 3, 3

We get a branch in our diagram, leading to two different evaluations.

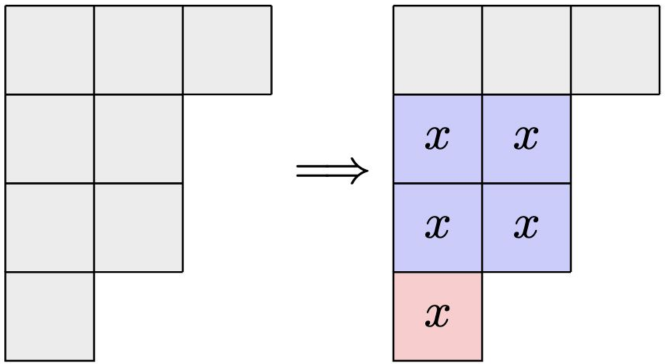
# 4

# Young Diagrams & Test Operations

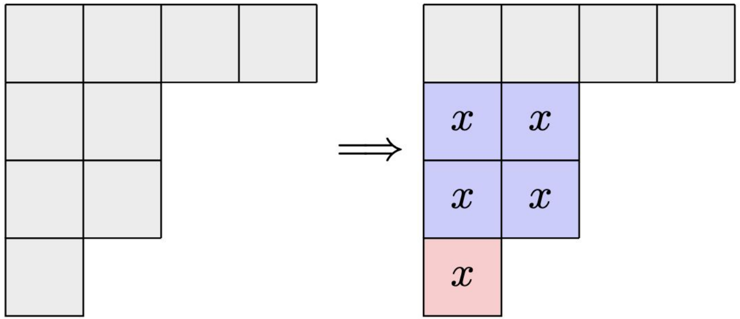
# Connection to Young Diagrams

We can connect Young Diagrams to 2 of our tests,

1, 2, 2, 3




1, 2, 2, 4



# Mitosis

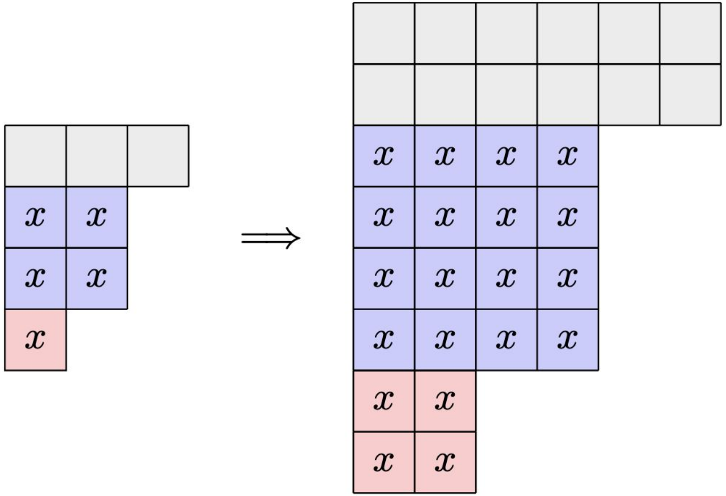
**Mitosis** is a way to multiply a test by a scalar such that the properties relating to the validity of its answers are preserved.

Multiply all of the answer choices, and the number of answer choices by the factor of mitosis.

**Example**: $(1, 2, 2, 3) \times 2 \Rightarrow 2, 2, 4, 4, 4, 4, 6, 6$

# Mitosis on Young Diagrams

Mitosis on Young Diagrams dilate the Young Diagrams by the scalar. For example, take the mitosis from before. In Young Diagrams it gives the following:



# 5

# Generating Functions & Sequences

# Number of k-solvable tests

The number of $k-\mathrm{solvable}$ tests has a generating function, where $s(n,k)$ is the number of $k-\mathrm{solvable}$ tests with a cost of $n$.

$$S_k(x) = \sum_{n=0}^{\infty} s(n,k)x^n = \prod_{j=1}^{\infty}\left(\frac{1}{1-x^j} - x^{j^2}\right) \sum_{1\leq i_1<\cdots<i_k} \prod_{r=1}^{k} \frac{x^{i_r^2}(1-x^{i_r})}{1-x^{i_r^2}+x^{i_r^2+i_r}}.$$

# Sequence for number of k-solvable tests

| $k\backslash n$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 1 | 0 | 1 | 2 | 1 | 3 | 6 | 5 | 10 | 11 | 18 | 21 | 32 | 38 | 54 | 67 | 88 | 112 | 152 | 178 |
| 2 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 1 | 1 | 2 | 2 | 3 | 4 | 4 | 7 | 10 | 14 | 15 | 19 | 29 |
| 3 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 1 | 1 | 1 |
| 4 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |

# Number of solvable tests

The number of solvable tests is:

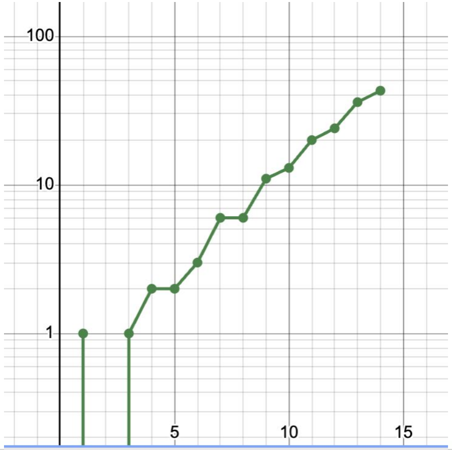
$$S_{>0}(x) = \sum_{n=0}^{\infty} s_{>0}(n)x^n = \prod_{k=1}^{\infty} \frac{1}{1-x^k} - \prod_{k=1}^{\infty} \left( \frac{1}{1-x^k} - x^{k^2} \right)$$

Which we count by calculating the number of partitions subtracting the number of unsolvable test.

# Sequence for solvable tests

We created an OEIS sequence for the number of solvable tests (A392745)

| $n$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $s(n)$ | 0 | 1 | 0 | 1 | 2 | 2 | 3 | 6 | 6 | 11 | 13 | 20 | 24 | 36 | 43 |



# Number of unsolvable tests

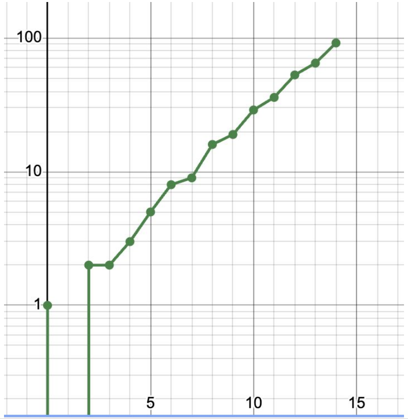
Now, this give the formula for the number of unsolvable tests.

$$S_0(x) = \sum_{n=0}^{\infty} S(n,0)x^n = \prod_{k=1}^{\infty} \left( \frac{1}{1-x^k} - x^{k^2} \right)$$

# Sequence of unsolvable tests

Using the generating function from the previous slide, we get the sequence for number of unsolvable tests (A276429)

| $n$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $s_0(n)$ | 1 | 0 | 2 | 2 | 3 | 5 | 8 | 9 | 16 | 19 | 29 | 36 | 53 | 65 | 92 |



# Acknowledgements

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# THANK YOU

# Works Cited

[1] Ivo David, Tanya Khovanova, and Yogev Shpilman, Mathematical Puzzles and Curiosities, (2025), World Scientific.