Down the Rabbit Hole of Vouchers

Presented by the PRIMES STEP Junior Group:

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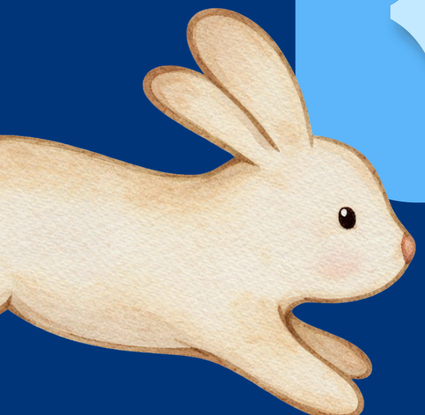
With thanks to our mentor, Tanya Khovanova



At the Self-Referential Store

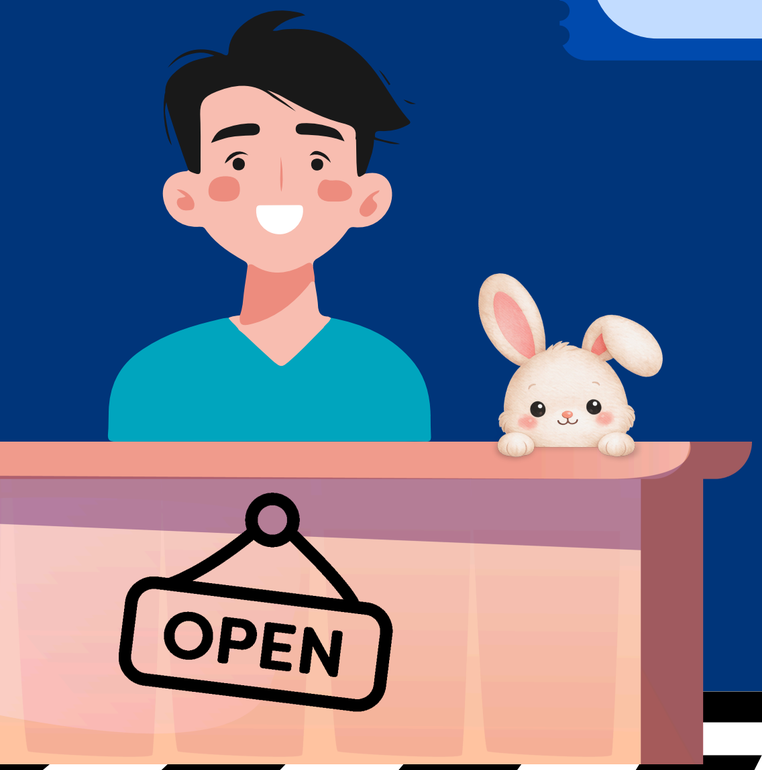
“If I buy a voucher worth $1, I pay $1.

Then, if I buy a voucher worth $2, I pay $2.

Then, if I buy a voucher worth $3, I pay...”

SIX DOLLARS?

Sorry, that's how it works.



1 2 3

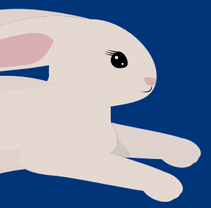
multiplies by 1 multiplies by 2 multiplies by 3

$1 + $2 + $6 = $9

TYPES OF COSTS

Voucher, Pairwise, and Loop

NOTATION



$\vec{v}$

The sequence of vouchers we buy.

v_i

v_i is the ith term in the sequence of vouchers $\vec{v}$.

$C_V(\vec{v})$

The voucher cost for a sequence of vouchers that we buy.

$C_P(\vec{v})$

The pairwise cost for a sequence of vouchers that we buy.

$C_L(\vec{v})$

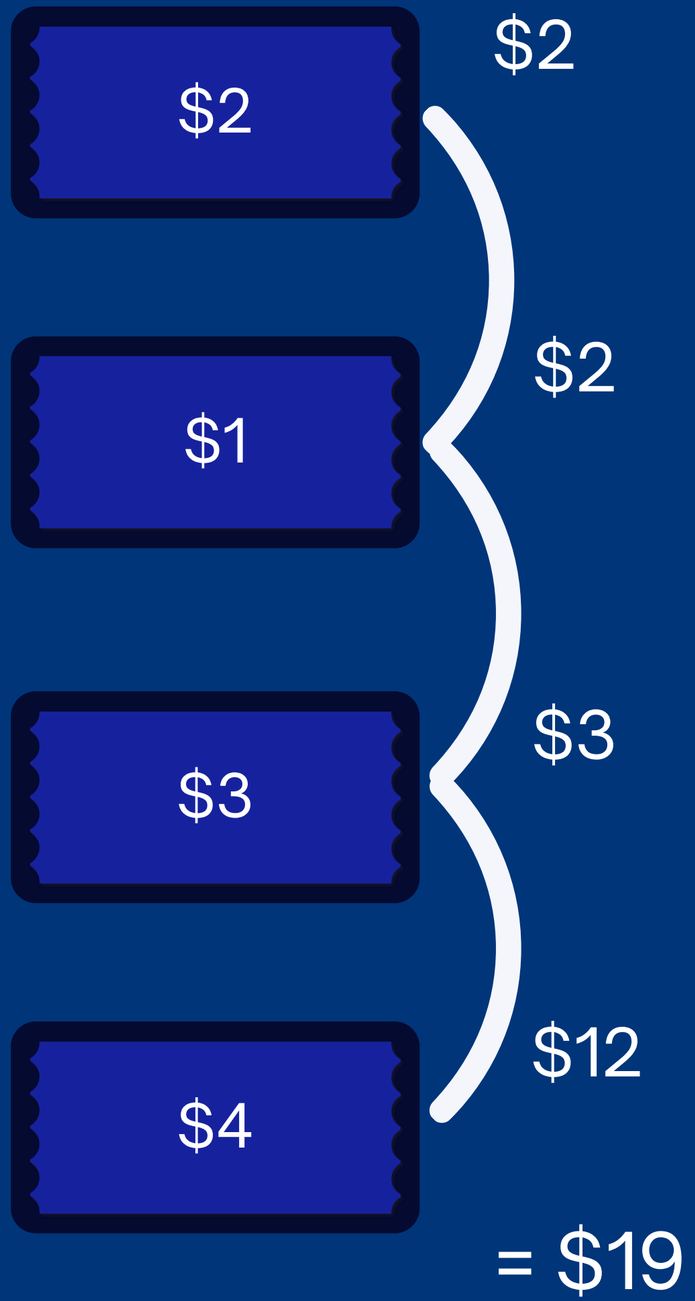
The loop cost for a sequence of vouchers that we buy.

VOUCHER

For calculating the voucher cost, we add the first value to the sum of the products of every adjacent pair of values.

$$1_{\text{st}} + \sum$$

$$C_V(\vec{v}) = v_1 + v_1 v_2 + \\ + v_2 v_3 + \cdots + v_{n-1} v_n$$



PAIR-WISE

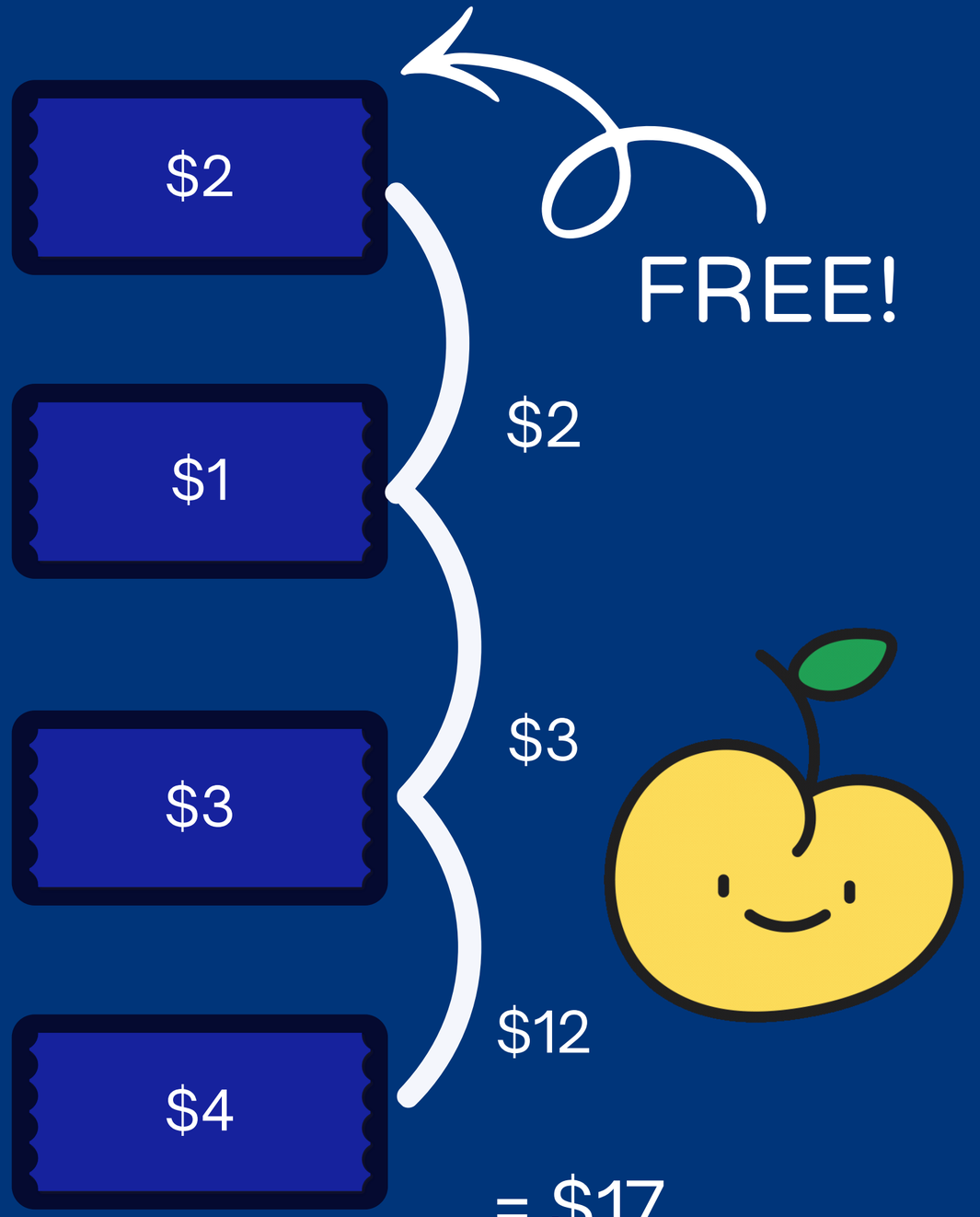
ALL SEASON SALE !

First voucher you buy is FREE!

However, multiplier still applies.

$$\sum$$ 

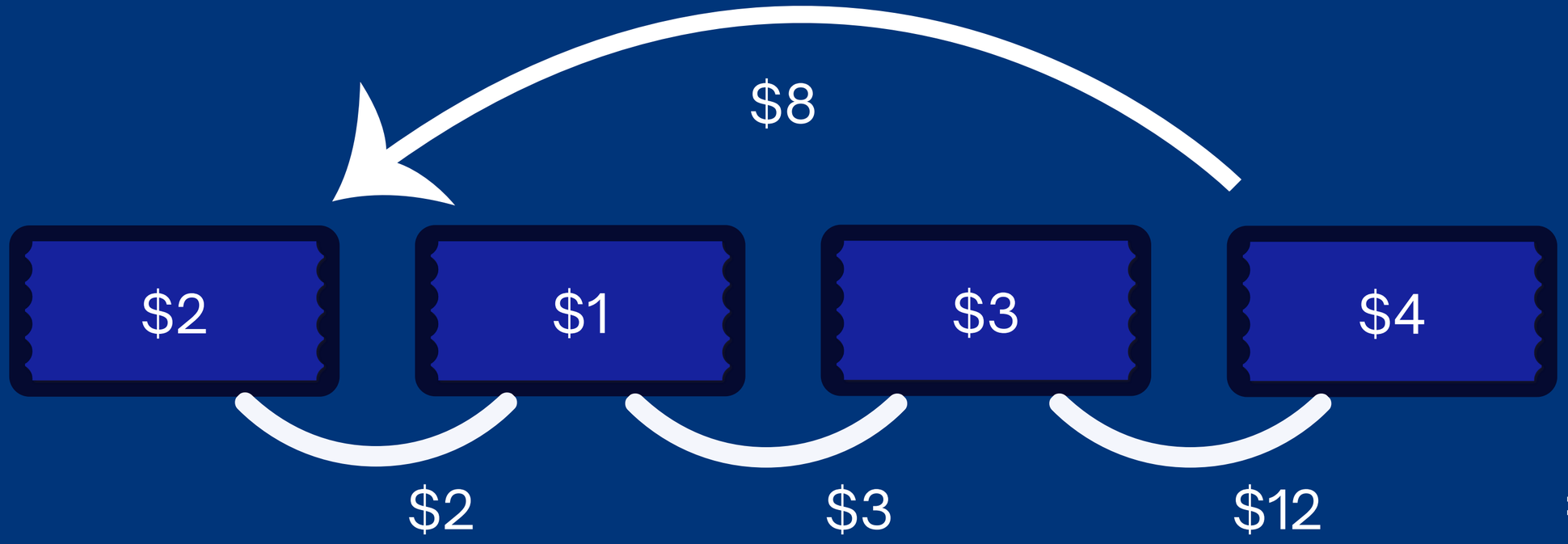
$$C_P(\vec{v}) = v_1v_2 + \\ + v_2v_3 + \cdots + v_{n-1}v_n$$



LOOP

The first one is free... but there is a fee!

BEWARE THE COST OF THE LAST VOUCHER!



$$C_L(\vec{v}) = v_1v_2 + v_2v_3 + \cdots + v_{n-1}v_n + v_nv_1$$

MINNIE

Wants to spend as little as possible but still needs vouchers.

Voucher: 2314 $15

Loop: 4231 $21

Pairwise: 3214 $12

MAX

Won the lottery, wants to show off his wealth by spending a lot.

Voucher: 3421 $25

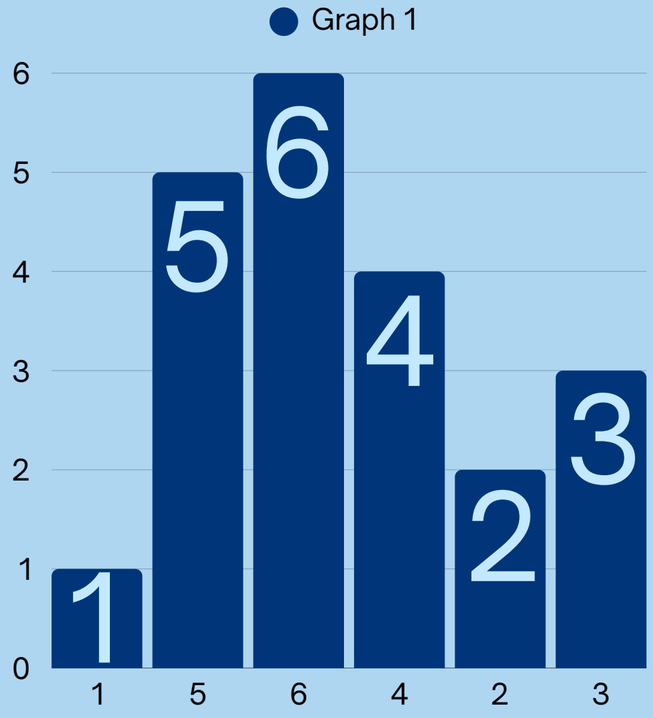
Loop: 4312 $25

Pairwise: 1342 $23

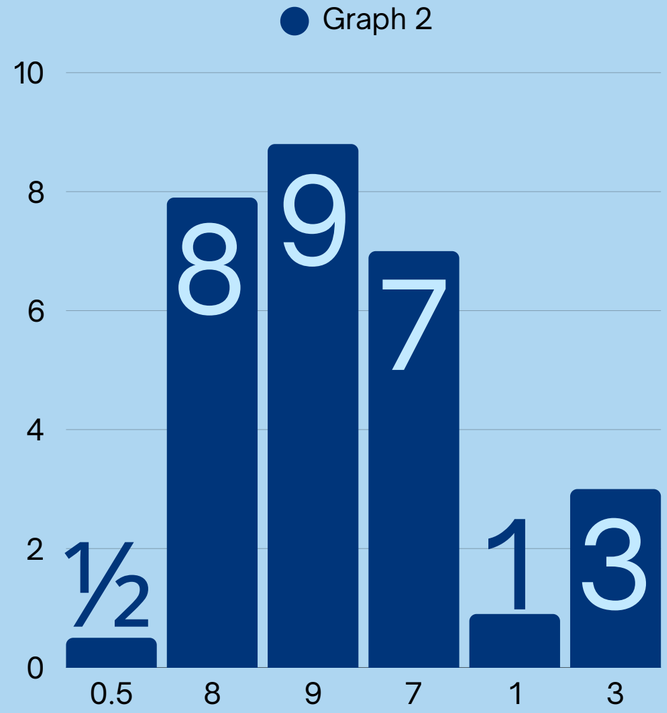
ORDER ISOMORPHISM

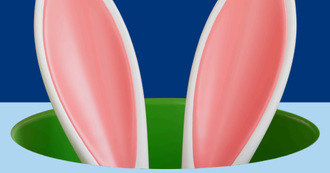
$1, 2, 3 = 2.4, e, \pi$

DEFINITION



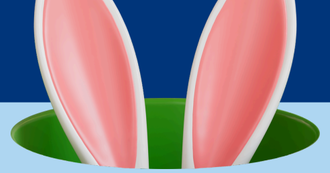
Order isomorphism means two ordered sequences are structurally identical, differing only in how their elements are labeled.





PROPERTIES OF ORDER ISOMORPHISM

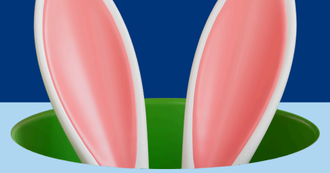
Theorem: *Any sequence that is order-isomorphic to a minimum or maximum sequence is a minimum or maximum sequence.*



PROPERTIES OF ORDER ISOMORPHISM

$$s_n = \text{nth smallest term}$$

- Simpler communication
  - 4, 2, 5, 1, 3 is easier to say than s_4, s_2, s_5, s_1, s_3
- All costs depend on it! (min/max of voucher, pairwise, loop)
- It retains minimum/maximum cost (less casework).
  - Ex: 2, 1, 3 realizes the minimum pairwise cost. Also e, 1, π achieves the minimum pairwise cost for those three vouchers.



PROPERTIES OF ORDER ISOMORPHISM

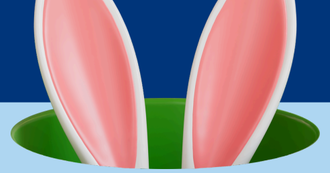
Maximum voucher and pairwise sequences for *n* are always order-isomorphic to*

$$2, 4, 6, 8, \ldots, n, \ldots, 7, 5, 3, 1$$

Minimum voucher and pairwise sequences for *n* are always order-isomorphic to*

$$n-1,\ 2,\ n-3,\ 4,\ \ldots,\ n-4,\ 3,\ n-2,\ 1,\ n$$

*only if numbers are greater than or equal to 1



PROPERTIES OF ORDER ISOMORPHISM

Maximum loop sequences for n is always order-isomorphic to

$$2, 4, 6, 8, \ldots, n, \ldots, 7, 5, 3, 1$$

up to rotations and reflections.*

Minimum loop sequences for n is always order-isomorphic to

$$n,\ 2,\ n-2,\ 4,\ \ldots,\ n-3,\ 3,\ n-1,\ 1$$

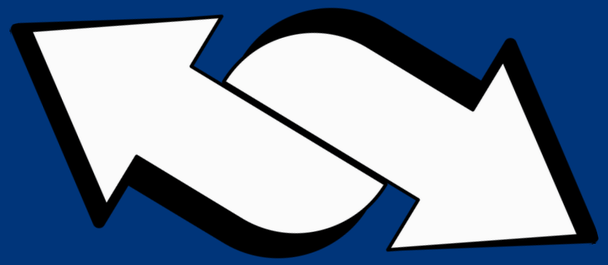
up to rotations and reflections*

*only if numbers are greater than or equal to 1

FLIPPING LEMMA

what if you could make things cheaper?

or, pricier?

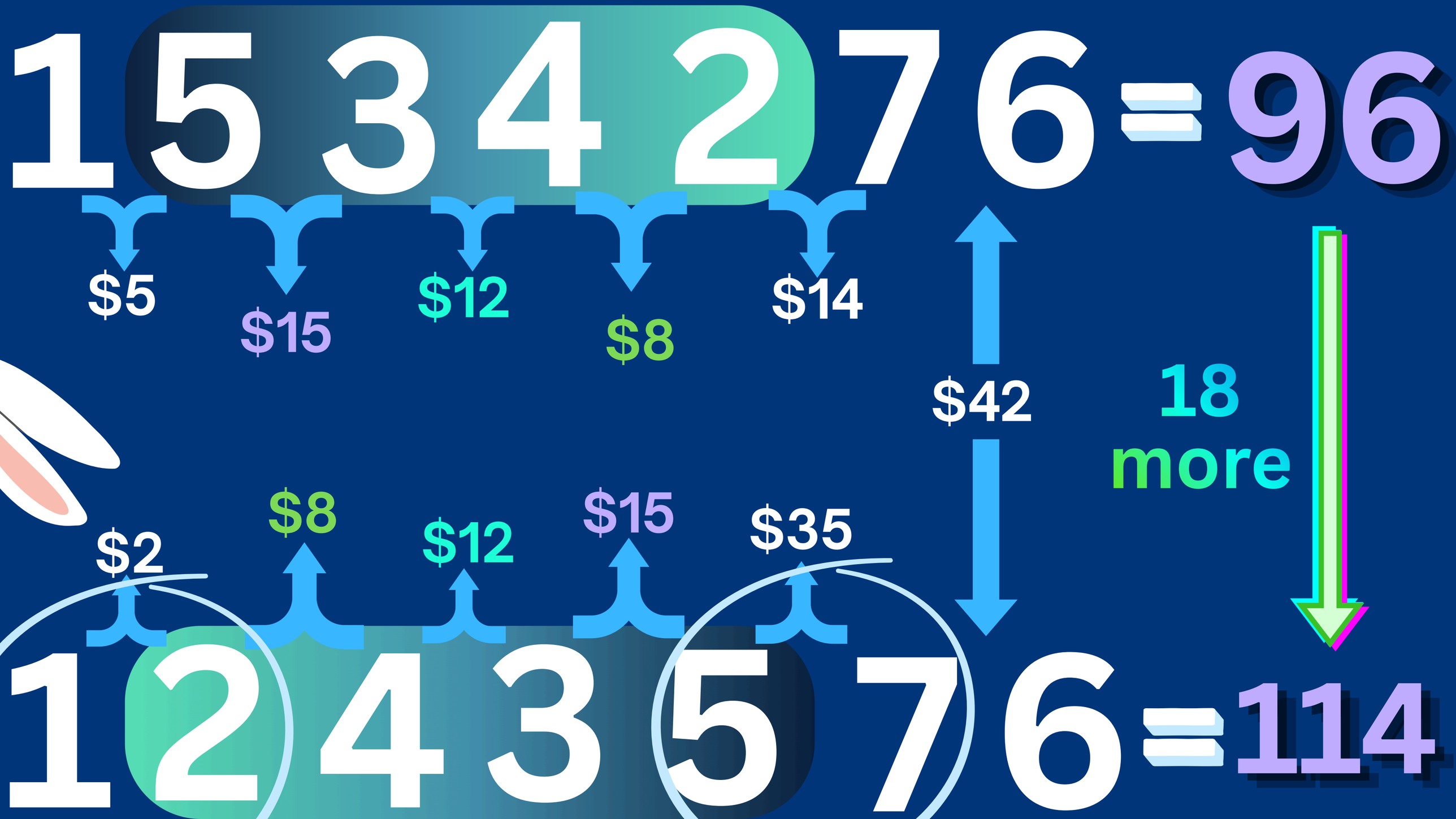


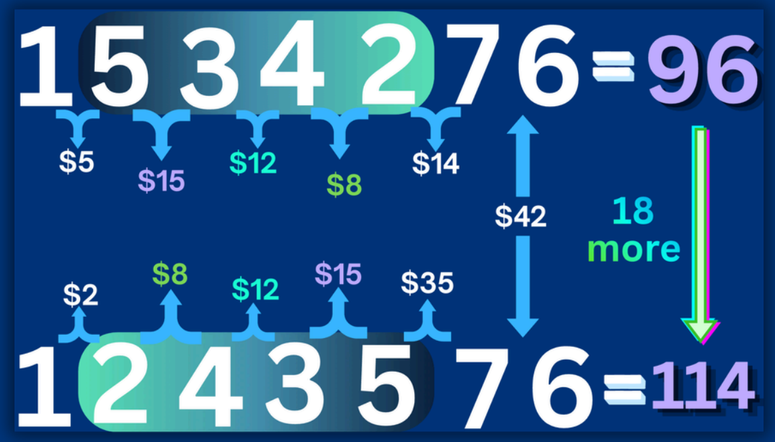
We flip the highlighted blue sequence.

12345 = 41

Only the circled areas change cost.

14325 = 33


1 5 3 4 2 7 6 = 96
$5
$15
$12
$8
$14
$42
18 more
$2
$8
$12
$15
$35
1 2 4 3 5 7 6 = 114



Why 18?

- Only boundaries of the flipped sequence change
- (2 + 35) - (5 + 14) = 18

Rearrangement Inequality!

Pairing big w/ big and small w/ small will cost more

EXAMPLE

Big with small

1*5 + 1*5 = 10

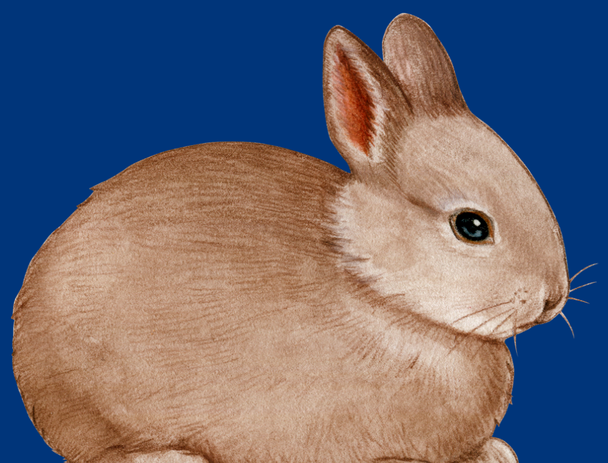
Big with big

1*1 + 5*5 = 26

$$ab + cd - ac - bd = (a - d)(b - c)$$

BOUNDARY RESPECTING PROCEDURES

How do we build the optimal order?



MAXIMUM VOUCHER BOUNDARY

- **Min-to-min boundary respecting construction**
- Set the left boundary to 1 and the right boundary to 0 (note: this varies depending on the type of cost)
- Start with the smallest voucher, which goes next to the 0 boundary
- Attach the next smallest term to the smaller boundary term
- Repeat until all the vouchers are placed
- This procedure gives a sequence with the maximum cost

sequence

1 3 7 5 2 0

Vouchers Left:

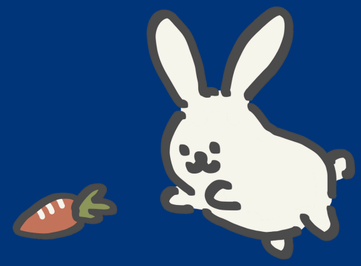
0.2 2 4 5

1

Left

0

Right

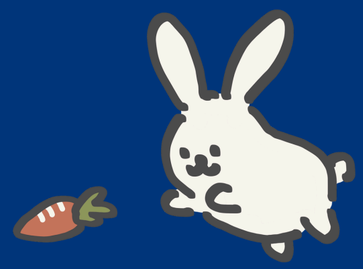


Vouchers Left:

2 4 5

1 0.2 0

Left Right

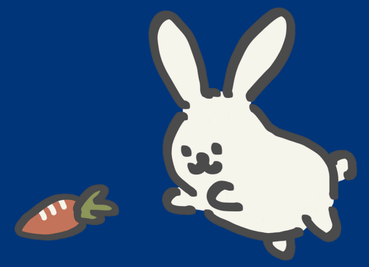


Vouchers Left:

4 5

1 2 0.2 0

Left Right

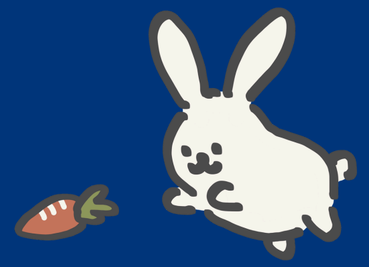


Vouchers Left:

5

1 4 2 0.2 0

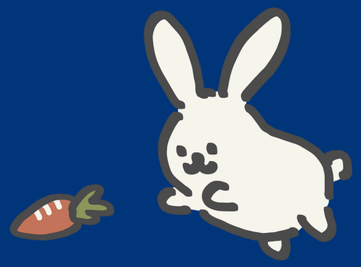
Left Right



Vouchers Left:

1 4 5 2 0.2 0

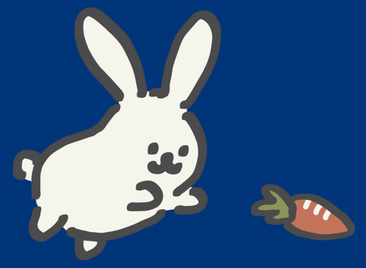
Left Right



Final Order:

4 5 2 0.2

Maximum Cost: $34.40



MINIMUM VOUCHER BOUNDARY

- **Max-to-min and min-to-max boundary respecting construction**
- Set the left boundary to 1 and the right boundary to 0 (note: this varies depending on the type of cost)
- Start with the largest voucher, which goes next to the 0 boundary
- Attach the smallest voucher to the largest boundary and then attatch the largest voucher to the smallest boundary
- Repeat until all the vouchers are placed
- This procedure gives a sequence with the minimum cost

sequence

1 5 3 2 7 0

Vouchers Left:

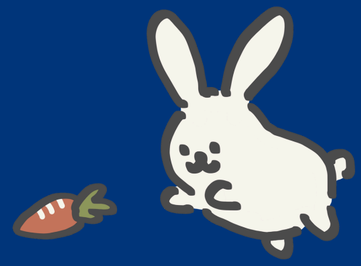
0.2 2 4 5

1

Left

0

Right

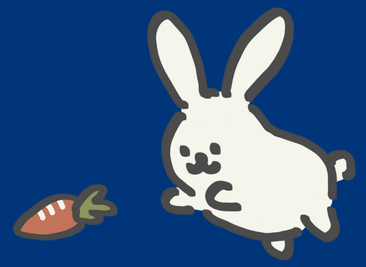


Vouchers Left:

0.2 2 4

1 5 0

Left Right

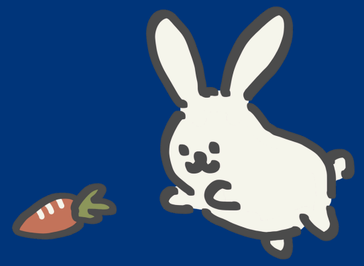


Vouchers Left:

2 4

1 0.2 5 0

Left Right

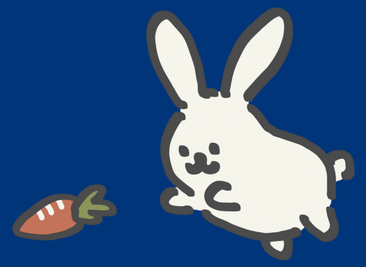


Vouchers Left:

2

1 4 0.2 5 0

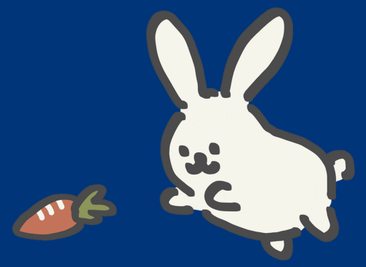
Left Right



Vouchers Left:

1 2 4 0.2 5 0

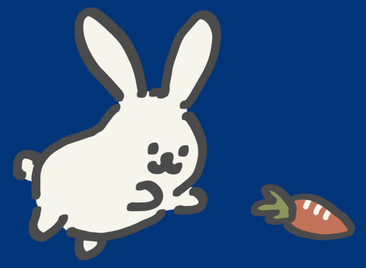
Left Right



Final Order:

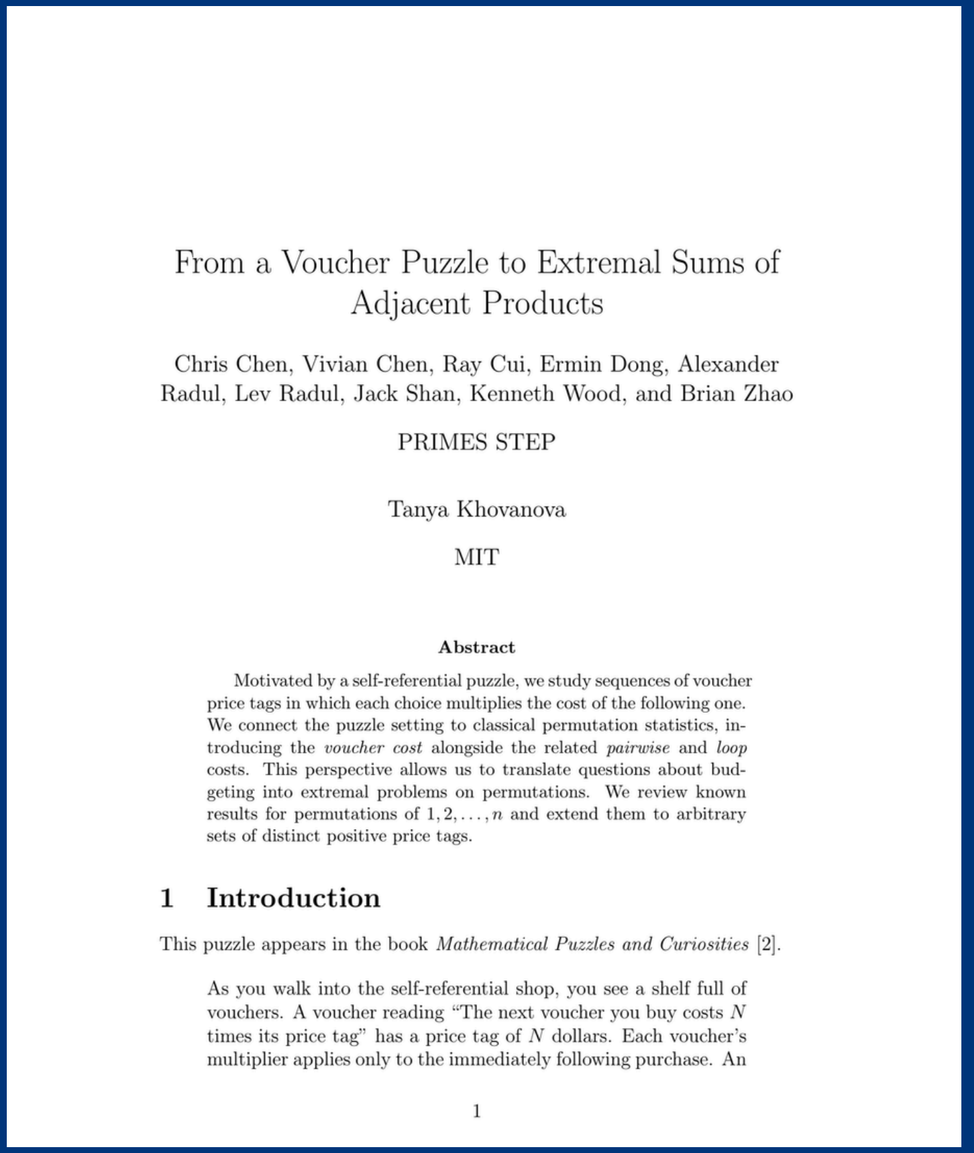
2 4 0.2 5

Minimum Cost: $11.80



OTHER STUFF WE DID

- Formulas for maximum and minimum voucher, pairwise, and loop costs
- Formulas for different sets of vouchers
  - Arithmetic Sequences
  - Geometric Sequences



From a Voucher Puzzle to Extremal Sums of Adjacent Products

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PRIMES STEP

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Abstract

Motivated by a self-referential puzzle, we study sequences of voucher price tags in which each choice multiplies the cost of the following one. We connect the puzzle setting to classical permutation statistics, introducing the *voucher cost* alongside the related *pairwise* and *loop* costs. This perspective allows us to translate questions about budgeting into extremal problems on permutations. We review known results for permutations of $1, 2, \ldots, n$ and extend them to arbitrary sets of distinct positive price tags.

1 Introduction

This puzzle appears in the book *Mathematical Puzzles and Curiosities* [2].

As you walk into the self-referential shop, you see a shelf full of vouchers. A voucher reading "The next voucher you buy costs N times its price tag" has a price tag of N dollars. Each voucher's multiplier applies only to the immediately following purchase. An

	Voucher	Pairwise	Loop
Minimum	$\frac{n^3+8n}{6} - \frac{15-3(-1)^n}{12}.$	$\frac{2n^3+4n-3+3(-1)^n}{12}$	$\frac{n^3+3n^2+5n}{6} - \frac{9+3(-1)^n}{12}$
Maximum	$\frac{2n^3+3n^2-11n+18}{6}.$	$\frac{2n^3+3n^2-11n+6}{6}$	$\frac{2n^3+3n^2-11n+18}{6}.$

ANY QUESTIONS?