How to (Potentially) Win a Million Dollars: An Introduction to the Riemann Zeta Function

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Outline

The Riemann Zeta Function

A millennium problem is a problem with a **one million dollar** prize. Today, we'll be covering the Riemann zeta function. Formally, it is defined as:

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The Riemann zeta function

For $\Re(s) > 1$,

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

This problem has puzzled mathematicians for centuries. It brings up questions such as:

- How is it connected to primes?
- What do you get when you plug in specific values?
- Where are its **nontrivial zeros?**

What are primes?

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Definition

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$$2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31 \ldots$$

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Can you see a pattern?

The Problem

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Theorem (Euclid's Theorem)

The number of primes is infinite.

Since there are **infinitely many** primes, and gaps between them are arbitrarily large, then is there even a pattern?
This brings us to our **motivating problem**:

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Problem

If there is a pattern to primes, how do we map their distribution?

The Gamma Function

Let's take a look a what it really means through factorials:
How do we compute factorials?

$$3! = 3 \times 2 \times 1 = 6$$

Simple enough, but what is 2.5! or 10.67! ? Let's try 2.5! is it...

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The Gamma function helps us compute non-integer factorials and even extends to the complex plane!

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$$\Gamma(s) = \int_0^\infty e^{-x} x^{s-1}\, dx \text{ for } s > 0.$$

Lemma

The (complete) gamma function $\Gamma(n)$ is defined to be an extension of the factorial to complex and real number arguments. It is related to the factorial by Γ(n)=(n-1)!

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So going back to our earlier examples $2.5! \approx 3.32$ and $10.67! \approx 17912460.77$

Bernoulli Numbers

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Can we save even more time by creating a **general formula** for calculating the sums of **all** powers?

Bernoulli Numbers

Definition

The Bernoulli numbers B_n are a sequence of signed rational numbers that can be defined by the exponential generating function

$$\frac{x}{e^x - 1} = \sum_{n=0}^{\infty} \frac{B_n x^n}{n!}.$$

They can also be defined as:

$$\frac{x}{e^x - 1} = B_0 + B_1 x + B_2 \frac{x^2}{2!} + B_3 \frac{x^3}{3!} + \cdots$$

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In short, the values of B_0, B_1, B_2, $B_3 \cdots$ are our Bernoulli numbers according to this function.

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Bernoulli Number Values for B_0 to B_6

$$B_0 = 1, B_1 = -\frac{1}{2}, B_2 = \frac{1}{6}, B_3 = 0, B_4 = -\frac{1}{30}, B_5 = 0, B_6 = \frac{1}{42}$$

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Notice a pattern?

Every odd Bernoulli number besides B_1 is 0.

Fun Fact

Bernoulli claimed that he could find the sum of the tenth powers of the first 1000 numbers in less than 7.5 minutes.

Convergence & Divergence

Definition

For any series $\sum a_n$, The series is *convergent* and *converges* to S if for all $\epsilon > 0$ there exists an N such that

$$\left| \sum_{k=1}^{n} a_k - S \right| < \epsilon$$

for all $n > N$. If no such S exists, the series diverges.

Notably, $\zeta(s)$ **converges** for $s > 1$.

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More simply, we can express it as:

$$\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} \dots$$

Will this sum **converge** or **diverge**?

Harmonic Series Proof

Proof.

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$$N \geq 1 + \frac{1}{2} + \underbrace{\frac{1}{4} + \frac{1}{4}} + \underbrace{\frac{1}{6} + \frac{1}{6}} + \underbrace{\frac{1}{8} + \frac{1}{8}} + \dots$$

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$$= \frac{1}{2} + N$$

This is a contradiction because $0 \neq \frac{1}{2}$, so the series **diverges**. □

The Basel Problem

The Basel Problem was posed in 1644 by Pietro Mengoli and eluded mathematicians for a century until Euler solved it in 1734.

Theorem (The Basel problem)

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

Notably, this is equivalent to $\zeta(2)$! By solving this, Euler was able to solve even convergent sums for the Riemann zeta function.

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$$\frac{\sin x}{x} = \left(1 - \frac{x}{\pi}\right)\left(1 + \frac{x}{\pi}\right)\left(1 - \frac{x}{2\pi}\right)\left(1 + \frac{x}{2\pi}\right)\dots$$

Do you notice a nice pattern in our equation?

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Proof.

Difference of squares!

$$\left(1-\frac{x^2}{\pi^2}\right)\left(1-\frac{x^2}{4\pi^2}\right)\left(1-\frac{x^2}{9\pi^2}\right)\cdots$$

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The x^2 coefficient for $\frac{\sin x}{x}$ above was $-\frac{1}{3!}=-\frac{1}{\pi^2}\left(\sum_{n=1}^{\infty}\frac{1}{n^2}\right)$

A Proof of the Basel Problem

Proof.

And finally, rearranging the terms we get:

$$\frac{\pi^2}{6} = \sum_{n=1}^{\infty} \frac{1}{n^2}$$

□

Finding all Even Values of ζ

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Theorem (Euler, 1739-1748)

$$\zeta(2k) = \frac{(-1)^{k+1} B_{2k} (2\pi)^{2k}}{2(2k)!}$$

Why is this so significant?

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Why is this so significant?

- We can find the exact values of all evens for $\zeta(s)$
- Proves that all even values of $\zeta(s)$ are irrational because of π
- Notice how the coefficient B_{2k} (A Bernoulli Number!) appears in this formula

Euler's Product Formula

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$$\sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{\text{prime } p} \frac{1}{1 - \frac{1}{p^s}}$$

Notice anything familiar? The left side is our Riemann zeta function!

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Notice anything familiar? The left side is our Riemann zeta function!
The right side is derived from our fundamental theorem of arithmetic - the prime factorization theorem.

Infinite Primes Proof

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$$\sum_{n=1}^{\infty} \frac{1}{n} \text{ diverges to } +\infty.$$

Thus,

$$\lim_{s \to 1^+} \zeta(s) = +\infty$$

Since each of the quantities $(1 - p^{-s})^{-1}$ stays bounded at $s = 1$, the divergence of the harmonic series implies that there are **infinitely many** primes. □

The Riemann Hypothesis

Let's start off with the following definition:

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With Z denoting the nontrivial zeros of the Riemann zeta function, Riemann hypothesized that:

The Riemann Hypothesis

$$Z \subset \{s \in \mathbb{C} \mid \Re(s) = 1/2\}$$

What's so significant about the Riemann hypothesis?

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Assuming 1 is **not prime**:

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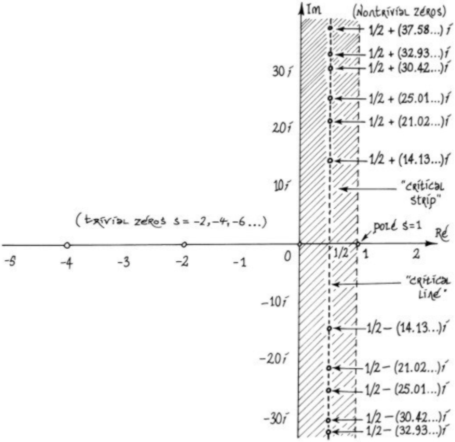
Logarithmic Integral Function

$$\mathrm{Li}(x) = \int_2^x \frac{dt}{\log t}$$

Zeros of the Riemann Zeta Function Visualized



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Thus,

Theorem

$$Li(x) \sim \pi(x) \; as \; x \to \infty$$

Error Bounds for Each Function

Without the Riemann hypothesis, the error bound for the prime counting function would look something like this:

Error Bound Without the Riemann Hypothesis

$$\pi(x) = \mathrm{Li}(x) + O(xe^{-\sqrt{\log(x)}})$$

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With the Riemann Hypothesis

$$\pi(x) = \text{Li}(x) + O(\sqrt{x}(\log(x))^2)$$

This allows us to get a greater estimate on the distribution of primes!

History

Here's a quick timeline of our problem:

300 BC	Euclid proves that there are an infinite number of primes
1644	Basel Problem proposed
1734	Euler solves the Basel Problem
1859	Riemann Hypothesis proposed
1896	Prime Number Theorem proven
2000	Riemann Hypothesis declared a millennium problem

How to Make a Million Dollars

So, how will you make a million dollars? You will simply have to prove that the Riemann hypothesis is true!
In today's talk, we talked about ideas such as

- the Gamma function
- Bernoulli Numbers
- the Prime Counting function
- the Logarithmic Integral Function
- Euler's product formula

A proof of the Riemann hypothesis would impact RSA encryption, affect how we see other millennium problems (such as P vs NP), and alter hundreds of theorems in number theory.
Hopefully, one of you can solve it in our lifetimes.

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- **Family:** We would also like to thank our families for always supporting us and our math progress
- **Audience:** Thank you for listening to our talk! Any questions?