Transformations in Geometry

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What We'll Cover Today

A look at the topics we will cover

Introduction

The history and importance behind Transformations

Understanding Functions & Maps

An explanation on functions and mapping from domain to codomain

Transformations

How Transformations affect geometric figures and applications

Connecting to Linear Algebra

Connecting transformational geometry to linear algebra

Introduction

What is Transformational Geometry?

Transformational Geometry

Transformational Geometry is a modern approach to geometry that focuses on studying figures by manipulating characteristics about them such as, position and size using functions.

History

Although moving geometric figures around is an ancient approach to geometry, the focus more recently shifted to a more function-based approach. As a result of the study of polynomials in the early nineteenth century, algebraic transformations and groups emerged.

Importance

It wasn't until early in the twentieth century that physicists realized the power of transformations, starting with Einstein's theory of relativity and then with quantum mechanics.

Understanding Functions and Maps

Characteristics of Sets and examples

Definition of Sets

Before introducing functions, we will first go over **sets**. Sets are collections of distinct objects, in the scope of this presentation, numbers. $N = \{1,2,3\}$.

Important Characteristics of Sets

- The order in which we write elements doesn't matter. $\{0,1\} = \{1,0\}$
- There are no repeated elements in a set. ✗ $\{0,1,2,3,3\}$
- To prove that two sets are equal, it needs to be proven that they share the same elements.
- A set can have no elements; this is called an empty set

Importance of Sets

Sets are fundamental concepts in Transformational Geometry as it is needed to define core concepts such as functions and transformation groups.

Understanding functions and maps

Explaining Functions & Injectivity and Surjectivity

Function Definition

In Transformational Geometry, **functions**

- are rules that map every figure (input) to a new location (output)
- often times changes qualities of the figure
- are defined by their name, followed by the input set (domain), and then the output set (codomain).

Example

f: R -> R, squares the domain and both the domain and codomain contain all real numbers.

1. f: (3) -> 9 and f: (-3) -> 9.
   The function is not injective.

2. f: (x) -> -9, x=no solution.
   The function is not surjective.

Injectivity

f: (R^2 -> R^2)
f(x,y) = (x',y')

Means that each input has a unique output, in other words, no two different points in the domain can map to the same point in the codomain.

Surjectivity

f: (R^2 -> R^2)
$\forall (x',y') \in R^2, \exists (x,y) \in R^2$ such that $f(x,y)=(x',y')$

Means that every point in the codomain came from a point in the domain.

Transformations

Algebraic Representation

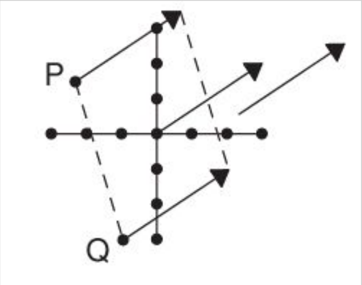
Definition of Transformations

Transformations are functions from R^2 -> R^2 that are bijective (both injective and surjective).

Translation

Translations slide every point of a figure the same distance in a specified direction.

$T(x,y) = (x+a, y+b) = (c,d)$



Source: Sibley, Thomas. Thinking Geometrically, p. 203

Identity

Identity transformation is unique as it does not change the input points.

$I(x,y) = (x,y)$

- Injective?
- Surjective?

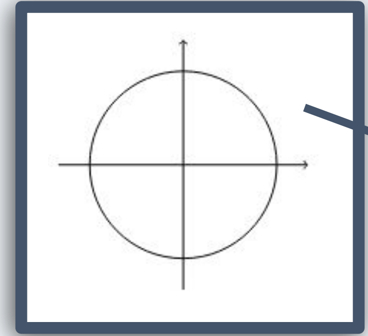
Important for inverse transformations since they revert the points back to they're starting position

Transformations

Applications

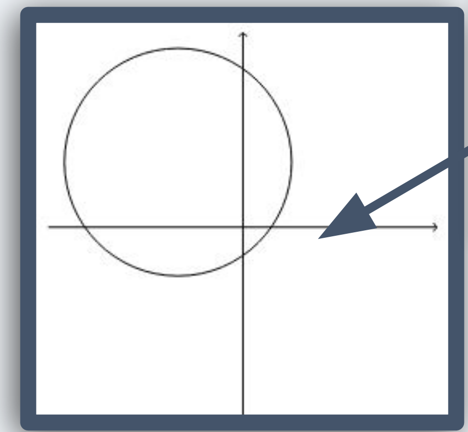
Unit Circle

- Has radius = 1
- Centered around (0,0)
- Equation: $1 = x^2 + y^2$



General Circle

- Has radius = R
- Centered around (a,b)
- Equation: $R^2 = (x-a)^2 + (y-b)^2$



Transformations

1. To change the radius from being exactly one to an arbitrary length, R: Dilate by a factor of R

2. To move the center of the unit circle to an arbitrary point (a,b): Translate by (+a, +b)

We can compose these two translations:
T o D(x,y) = T(D(x,y)) = T(Rx,Ry) = (Rx+a, Rx+b)

Connecting to Linear Algebra

Transformation Groups

Definition of a Group

A group is defined as containing a set and an operation and has to satisfy three conditions. Ex: (R, +)

Transformation Group

A transformation group is defined as a group with the set being a collection of transformations and the operation being the composition of them.

The transformations that dilated and translated the unit circle are considered a transformation group.

Closure

If any two elements in the set are combined using the operation, the result stays inside the set. Ex: 2 + 3 = 5

Identity

There exists a special element, e that leaves every other element unchanged when combined with it. Ex: 7 + 0 = 7

Inverses

There exists some method to arrive back at e from any element in the set. 8 + (-8) = 0

Connecting to Linear Algebra

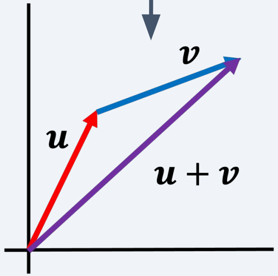
Vector Space

Vector

A vector is defined as a quantity with direction and magnitude, often depicted as an arrow. Vectors can be added or subtracted with one another.

$$\vec{a} = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$$

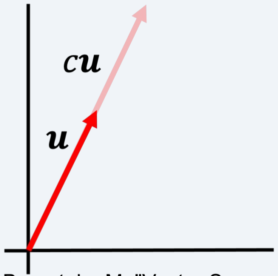
$$\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} x_1 + x_2 \\ y_1 + y_2 \end{pmatrix}$$



Source: Bernstein, M. "Vector Spaces,"

Vector Space

A vector space, V, is a group with the set of all vectors, S and the operation being addition. V= (S, +)



Source: Bernstein, M. "Vector Spaces,"

$$\lambda\vec{a} = \lambda \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} \lambda x_1 \\ \lambda y_1 \end{pmatrix}$$

Scalar Multiplication

On top of this, in the Vector Space, there is scalar multiplication, where every vector can be multiplied with a scalar (real number) .

Connecting to Linear Algebra

Linear Independence

Linear Combination + Linear Independence

Linear Combination
For the set of vectors S, we can get the **linear combination** which is the sum of the vectors and each vector is multiplied by some scalar.

$$\lambda_1 \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + \lambda_2 \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \dots + \lambda_n \begin{pmatrix} x_n \\ y_n \end{pmatrix}$$

Linearly Independent:
When the only solution to the linear combination equaling 0 is when the scalars all equal 0, then the set is **Linearly Independent**.

The importance of this is when defining the basis of a vector space.

Linearly Independent

For instance, let's say the set of vectors S = $\left\{ \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} -3 \\ 1 \end{pmatrix} \right\}$
If we were to set the linear combination to 0 we would get :

$$\lambda_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} + \lambda_2 \begin{pmatrix} -3 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

In this case, only when $\boldsymbol{\lambda_1}, \boldsymbol{\lambda_2} = 0$ then the linear combination is 0.

Linearly Dependent

For instance, let's say the set of vectors S = $\left\{ \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 4 \end{pmatrix} \right\}$
If we were to set the linear combination to 0 we would get :

$$\lambda_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} + \lambda_2 \begin{pmatrix} 2 \\ 4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

In this case, when $\boldsymbol{\lambda_1} = -2$ and $\boldsymbol{\lambda_2} = 1$ then the linear combination is 0.

Connecting to Linear Algebra

Spanning Set & Basis

Spanning Set

For every set of vectors S we can also determine if it **spans** the Vector Space V. This means that for every vector that is an element of V can be written as a linear combination of S.

$\mathrm{span}(v_1, v_2, \ldots, v_n) = V$

Referring back to the previous example, to the set of vectors S= $\left\{ \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} -3 \\ 1 \end{pmatrix} \right\}$

This is considered spanning for the Vector Space V defined as $(R^2, +)$

Basis

The **basis** of the vector space is the set of vectors S that is both linearly independent and spanning.

There is also the **canonical basis** which is the simplest way of writing the basis.

For the Vector Space V defined as $(R^2,+)$ the canonical basis would be: $\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$

Connecting to Linear Algebra

Using Matrices to Transform Vectors

Matrices

+ - X

In transformational geometry, **matrices**, which are rectangular arrays of numbers, can be used to represent a transformation.

- The values inside the matrix are based on the mappings of the canonical basis.

$$M\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax + by \\ cx + dy \end{pmatrix}$$

Identity

If we were to have the matrix be the exact same values of the canonical basis then the transformation would make no changes on the vector.

$$M\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x + 0 \\ 0 + y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$

This is an **identity** transformation which we examined earlier.

Dilation

Consider, if we were to have the matrix be the the same values of the canonical basis but dilated by 3.

$$M\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3x + 0 \\ 0 + 3y \end{pmatrix} = \begin{pmatrix} 3x \\ 3y \end{pmatrix}$$

This is a **dilation** which we also examined earlier.

Thank You

Any Questions?