Setting up Set Theory!

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The Origins of Set Theory

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- $\{1, 2, 3\} \neq \{4, 5, 6\}$
- $1 \in \{1, 2, 3\}$

George Cantor and Naive Comprehension

In the 1890s, the mathematician George Cantor first formulated the foundations of set theory. His ideas are known as Naive Comprehension, which means that any set can be constructed given a certain condition. Which notation is as such

$$\{x \mid P(x)\}$$

For example, the condition could be the set of all natural numbers that are even, which can be written like this

$$\{x \in N \mid x \text{ is even}\}$$

However, this unrestricted version of set theory leads to contradictions and paradoxes such as Russell's, Burali-Forti's, and Cantor's.

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- If there are n elements in a set, the power set contains 2^n elements

Cantor's Theorem

For every set A,

$$|A| < |P(A)|$$

In other words: the power set of A is always strictly larger than A.

Cantor's Paradox

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But Cantor's theorem says that

$$|A| < \mathcal{P}(A)$$

Russell's Paradox

In a letter to fellow mathematician Gottlob Frege, Bertrand Russell communicated the following paradox.

- He found an inconsistency within naive set theory based upon this idea $x|\varphi(x)$, meaning it is possible to construct a set of all things that follow a specific condition.
- This led Russell to consider what would happen when the condition is exclusivity
- Meaning the condition is that x does not include itself: $R = (x|x \notin x)$
- This leads to the contradiction, whether R is a member of itself.

Burali-Forti Paradox

Transitive Set

A set α is called transitive if

$$x \in \alpha \Rightarrow x \subseteq \alpha$$

In other words, every element of an element of α is also contained in α.

Example of Transitivity

$$2 = \{0, 1\} = \{\varnothing, \{\varnothing\}\}$$

Since

$$1 = \{\varnothing\} \in 2$$

and

$$1 \subseteq 2,$$

every element of 1 is also an element of 2. Thus 2 is transitive.

Ordinal An ordinal is a transitive set that is well-ordered by $\in$.

Examples:

$$0 = \varnothing, \qquad 1 = \{0\}, \qquad 2 = \{0, 1\}, \qquad 3 = \{0, 1, 2\}$$

Successor ordinal:

$$\alpha + 1 = \alpha \cup \{\alpha\}$$

Theorem There is no set of all ordinals.

Assume for contradiction that

$$\Omega = \{\alpha \mid \alpha \text{ is an ordinal}\}$$

exists.

Step 1: Ω is well-ordered Every nonempty subset of ordinals has an $\in$-least element. Therefore every nonempty subset of Ω has the least element. Hence,

$$\Omega \text{ is well-ordered by } \in .$$

Step 2: Ω is transitive If $\alpha \in \Omega$, then α is an ordinal. Every element of an ordinal is also an ordinal, so

$$\beta \in \alpha \Rightarrow \beta \in \Omega.$$

$$\alpha \subseteq \Omega.$$

Hence Ω is transitive. Since Ω is transitive and well-ordered, Ω itself is an ordinal. Therefore,

$$\Omega \in \Omega.$$

But no ordinal can contain itself. Equivalently,

$$\Omega + 1 = \Omega \cup \{\Omega\}$$

is a larger ordinal, contradicting the assumption that Ω already contains all ordinals.

Creating ZFC

- In response to all these contradictions, mathematicians sought to put some rules in place.
- The first attempt at axiomatization was by mathematician Ernst Zermelo in 1908
- Zermelo's work was later expanded upon by Abraham Fraenkel and Thoralf Skolem, resulting in what is now known as Zermelo-Fraenkel set theory (ZF), which forms the basis of most modern set theory.
- Later, the Axiom of Choice was added to Zermelo-Fraenkel so the system became ZFC, which to this day remains the agreed-upon foundation of mathematics.

Axioms of ZFC

- **Axiom of Existentiality**: If set X and set Y have the same elements, then they are equal
- **Axiom of Pairing**: For all elements a and b there exists a set a,b that contains only a and b
- **Axiom of Separation**: A set can only be created from an existing set
- **Axiom of Union**: For any set X, there exists a set that is the union of all the elements of X
- **Axiom of the Power Set**: For any X there exists a set Y = P(x), the set of all subsets of X
- **Axiom of Infinity**: There is an infinite set
- **Axiom of Replacement**: If there is a set and a clear rule to change each item of a set, then the result is also a set
- **Axiom of Foundation**: A set contains a smallest element.
- **Axiom of Choice**: Given a collection of non-empty sets, it is possible to choose one element from each set to make a new set, even if there is no rule in how you make those choices.

Resolving Russels Paradox with ZFC

$$\{x \in A \mid P(x)\}$$

Using ZFC's Axiom of Separation, which states that sets can only be created from an existing set, and since ZFC does not account for a Universal set (set of all sets), Russell's paradox cannot be created.

Resolving Cantor's Paradox with ZFC

Cantor's theorem remains true in ZFC:

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What ZFC rejects is the naive step that:

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Because there is no set of all sets, Cantor's theorem cannot be applied to a universal set V.

Resolving Burali–Forti Paradox with ZFC

$$\Omega = \{\alpha \mid \alpha \text{ is an ordinal}\}$$

According to the Axiom of Separation,

$$\{\alpha \in A \mid \alpha \text{ is an ordinal}\}$$

- Sets can only be formed from an already existing set
- There is no set containing all ordinals
- Therefore, the collection of all ordinals cannot be constructed as a set

Future Directions of ZFC

- ZFC is the standard foundation of modern mathematics
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Von Neumann - Bernays - Godel (NBG) Set Theory

- It extends ZFC by formally including classes
- It distinguishes between: sets and proper classes
- It allows working with objects like the class of all sets and the class of all ordinals

Why We Care About the Axiomatic Approach

By constructing axioms and taking an axiomatic approach in set theory:

- we can prevent paradoxes by avoiding inconsistencies
- we clarify what it means for something to 'exist' in math
- we can lay the foundation for comparisons between the mathematical world, depending on the axiom systems