PRIMES 2025 Entrance Problem Set

October 1, 2024

Notation: We let $\mathbb{N}$, $\mathbb{N}_0$, $\mathbb{Z}$, and $\mathbb{R}$ denote the sets of positive integers, nonnegative integers, integers, and real numbers, respectively.

Problem 1. Let $g\colon \mathbb{R} \to \mathbb{R}$ be a differentiable function satisfying the following conditions.

- $g(0) = 1$ and $g(t) \geq 0$ for all $t \in \mathbb{R}$.
- The derivative function $g'\colon \mathbb{R} \to \mathbb{R}$ is continuous.

Argue that the following inequality holds:

$$\left| \int_0^1 g(t)\, dt - \int_0^1 g(t)^3\, dt \right| \leq M \left(\int_0^1 g(t)\, dt \right)^2,$$

where M is the maximum value of $|g'(t)|$ in the closed interval $[0, 1]$.

Problem 2. Consider a chessboard of length 12 (with 144 unit squares). Two distinct unit squares of the chessboard are called **adjacent** if they share an edge. Find the largest $m \in \mathbb{N}$ such that whenever we mark the $2m$ unit squares covered by any m disjoint pairs of adjacent units, there are still two adjacent unit squares that remain unmarked.

Problem 3. For $n \in \mathbb{N}$, let $b = b_1 b_2 \dots b_n$ be a binary string with $b_1 + b_2 + \cdots + b_n \geq 1$ (that is, $b_i = 1$ for at least one index i), and let $v = (c_1, c_2, \dots, c_n)$ be a **value vector** with entries in $\mathbb{N}$. An improvement operation on the value vector v with respect to the binary string b consists of the following two steps.

1. Choose an index i from the set $\{1, 2, \dots, n\}$ with probability $\frac{c_i}{c_1 + c_2 + \cdots + c_n}$.
2. For a chosen index i, replace c_i by $c_i - 1$ if $b_i = 0$ and replace c_i by $c_i + 1$ if $b_i = 1$.

Design an efficient algorithm that computes the expected value of each entry of the vector value v after m improvement operations, and explain the worst time complexity of your algorithm in terms of m and n.

Problem 4. Let $\mathbb{F}_2$ be the field of two elements, and let $\mathbb{F}_2[x]$ be the ring of polynomials over $\mathbb{F}_2$. For each $n \in \mathbb{N}$, consider the following polynomial in $\mathbb{F}_2[x]$:

$$f(x) = x^{6n} + x^{5n} + x^{4n} + x^{3n} + 1.$$

(a) For which values of n, does $f(x)$ factor into exactly two irreducible polynomials in $\mathbb{F}_2[x]$?

(b) For which values of n, does $f(x)$ factor into exactly three irreducible polynomials in $\mathbb{F}_2[x]$?

Problem 5. The special linear group $SL(2,\mathbb{Z})$ over $\mathbb{Z}$ is the multiplicative group consisting of all 2×2 matrices with entries in $\mathbb{Z}$ and determinant 1; that is,

$$SL(2,\mathbb{Z}) = \left\{ \begin{pmatrix} a, b \\ c, d \end{pmatrix} : a, b, c, d \in \mathbb{Z} \text{ and } ad - bc = 1 \right\}.$$

Let G be the quotient group $SL(2,\mathbb{Z})/\{\pm I\}$, where I is the 2×2 identity matrix. Find with proof the number of subgroups of G of index m for each $m \in \{2, 3, 4, 5, 6\}$.

Problem 6. Let $\mathbb{N}_0[\![x]\!]$ be the set of all nonzero generating functions $\sum_{n=0}^{\infty} a_n x^n$ with coefficients in $\mathbb{N}_0$ (having at least one nonzero coefficient), and observe that $\mathbb{N}_0[\![x]\!]$ is closed under the standard multiplication of generating functions: for any $f(x) = \sum_{n=0}^{\infty} a_n x^n$ and $g(x) = \sum_{n=0}^{\infty} b_n x^n$ in $\mathbb{N}_0[\![x]\!]$,

$$f(x)g(x) = \left(\sum_{n=0}^{\infty} a_n x^n \right) \left(\sum_{n=0}^{\infty} b_n x^n \right) := \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} a_k b_{n-k} \right) x^n \in \mathbb{N}_0[\![x]\!].$$

- A generating function $f(x) \in \mathbb{N}_0[\![x]\!] \setminus \{1\}$ is called **indecomposable** if whenever the equality $f(x) = g(x)h(x)$ holds for some $g(x), h(x) \in \mathbb{N}_0[\![x]\!]$, either $g(x) = 1$ or $h(x) = 1$.
- A generating function $s(x) \in \mathbb{N}_0[\![x]\!]$ is called **supported** if either $s(x) = 1$ or $s(x)$ can be written as a product of finitely many indecomposable generating functions in $\mathbb{N}_0[\![x]\!]$ (repetitions of factors are allowed).

(a) Find a generating function in $\mathbb{N}_0[\![x]\!]$ that is not supported.

(b) Is it possible to find, for each $f(x) \in \mathbb{N}_0[\![x]\!]$, a supported generating function $s(x) \in \mathbb{N}_0[\![x]\!]$ such that $s(x)f(x)$ is also supported?

(c) Is it possible to find a supported generating function $s(x) \in \mathbb{N}_0[\![x]\!]$ such that $s(x)f(x)$ is a supported generating function for all $f(x) \in \mathbb{N}_0[\![x]\!]$?