Generalized Markov traces

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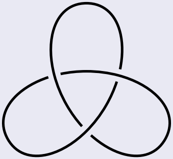
Knots and links

Definition

Knot is an embedding of a circle S^1 into $\mathbb{R}^3$. Link is a disjoint union of knots.



Braids and braid group

Definition

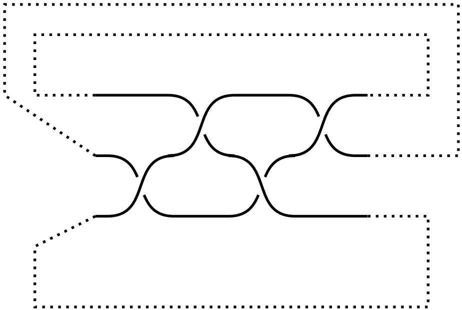
The braid on n strands is formed when n points on a horizontal line are connected by n strands to the n points on another horizontal line directly below, and where the strands descend all the time along the way.

Braid group Br_n is a group of braid equivalence classes under ambient isotopy. It is well-known that Br_n is a group on generators $\sigma_1, \ldots, \sigma_{n-1}$ subject to the braid relations:

- $\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$,
- $\sigma_i \sigma_j = \sigma_j \sigma_i$ for $|i - j| > 1$.

Braid closure

Braid β can be turned into a link by connecting the opposite nodes of β. This operation is called the closure of a braid β, and we will denote it by $\overline{\beta}$.



Coxeter groups

Definition

Coxeter group W is a group with presentation $\langle\, s_1, \ldots, s_n \mid (s_i s_j)^{m_{ij}} = 1 \,\rangle$ where $m_{ii} = 1\ \forall i = \overline{1, n}$ and $m_{ij} = m_{ji} \geq 2$ is an integer or ∞ for $i \neq j$.

Note that the symmetric group is a Coxeter group with $m_{i,i+1} = m_{i+1,i} = 3$ and the braid group is obtained if we forget the relations $s_i^2 = 1$.

Iwahori-Hecke algebra

Definition

The Iwahori-Hecke algebra $\mathcal{H}(W)$ is a unital algebra over a ring $\mathbb{Z}[v, v^{-1}]$ generated by the elements

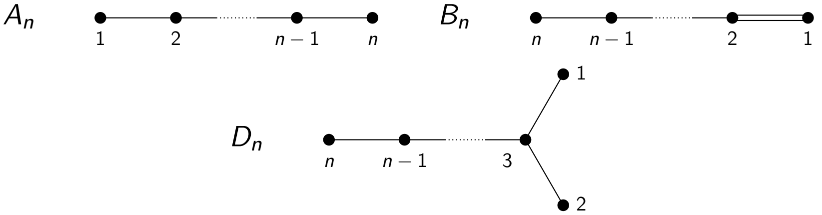
$$t_{s_i} := t_i, s_i \in S = \{s_1, \ldots, s_n\},$$

where S is the set of generators of W, satisfying the following relations:

$$t_i^2 = (v_i - v_i^{-1})t_i + 1,$$
$$\underbrace{t_i t_j \cdots}_{m_{ij}} = \underbrace{t_j t_i \cdots}_{m_{ij}}.$$

Embeddings of Dynkin diagrams

Dynkin diagram of a Coxeter group W is a graph with an adjacency matrix (m_{ij}). For simplicity, $2-$edges are omitted, $3-$edges are drawn as single edges and 4-edges are drawn as double edges.
Let $X_n = A_n,\ B_n, D_n$. For the inclusion of Dynkin diagrams $\Gamma_{n-1} \subset \Gamma_n$ shown below where $|\Gamma_n \setminus \Gamma_{n-1}| = 1$, we define an embedding $\iota : \mathcal{H}(X_{n-1}) \hookrightarrow \mathcal{H}(X_n)$.



Full twist and Jucys-Murphy elements

Define the canonical lift $w \mapsto \widetilde{w}$ from W to $\mathcal{H}(W)$ as follows: if $w = s_{i_1} \cdots s_{i_r}$ is a reduced expression, then $\widetilde{w} = t_{i_1} \cdots t_{i_r}$. Let $w_0 \in W$ be the longest element.

Proposition

The full twist $FT(W) = \widetilde{w_0}^2$ is central in $\mathcal{H}(W)$.

Let $J(X_n) = FT(X_n) \cdot FT(X_{n-1})^{-1}$ be the Jucys-Murphy elements in type X_n and $j(X_n) = t_{w_0,X_n} \cdot t^{-1}_{w_0,X_{n-1}}$.
$J(X_i)$ is a collection of commuting elements in $\mathcal{H}(W)$, playing an important role in its representation theory.

Markov trace

Theorem (Markov)

Given two braids $\beta_1, \beta_2 \in Br$, their closures are equivalent links if and only if β_2 can be obtained from β_1 by a sequence of the following moves:

- Conjugation of $\alpha \in Br_n$ in Br_n;
- Replacing $\alpha \in Br_n$ by $\alpha\sigma_n^{\pm 1} \in Br_{n+1}$.

This theorem inspires the following theorem/construction:

Markov trace

Let

$$T_n = t_n^{-1} \dots t_2^{-1} t_1 t_2 \dots t_n \in \mathcal{H}(B_n),$$
$$U_n = t_n^{-1} \dots t_3^{-1} t_1^{-1} t_2 t_3 \dots t_n \in \mathcal{H}(D_n)$$

Theorem (Jones [3], Geck-Lambropoulou [1])

There is a system of traces $Tr_{X_n} : \mathcal{H}(X_n) \to \mathbb{Q}[a, v_1, \dots, v_n, y]$, uniquely defined by the following relations:

- $Tr_{X_0}(1) = 1$,
- $Tr_{X_n}(xz) = Tr_{X_n}(zx)$,
- $Tr_{X_n}(\iota(x)) = (1 + a) Tr_{X_{n-1}}(x)$ for $x \in \mathcal{H}(X_{n-1})$,
- $Tr_{X_n}(\iota(x) t_n) = (v_n - v_n^{-1}) Tr_{X_{n-1}}(x)$ for $x \in \mathcal{H}(X_{n-1})$,
- $Tr_{B_n}(\iota(x) T_n) = y Tr_{B_{n-1}}(x)$ for $x \in \mathcal{H}(B_{n-1})$, if $X = B$.
- $Tr_{D_{2n}}(\iota(x) U_{2n-1} U_{2n}) = y^2 Tr_{D_{2n-2}}(x)$ for $x \in \mathcal{H}(D_{2n-2})$, if $X = D$.

Multivariable link invariant

The Markov trace given above can be modified to give a genuine link invariant.

Definition (*HOMFLY-PT* polynomial)

$$P(\overline{\beta}) = \frac{\sqrt{-a}^{e(\beta)}}{\left(\sqrt{-a}(v - v^{-1})\right)^{n-1}} Tr_{A_n}(\pi(\beta))$$

where $e(\beta)$ is the exponent sum of $\beta \in Br_n$, and the projection $\pi(\sigma_i) = t_i \in \mathcal{H}(A_{n-1})$.

Markov trace in type A

We have the following classical result.

Theorem

$$Tr_{A_n}(x) = \textit{coefficient near } 1 \textit{ in } \{t_w^{-1}\}_{w \in A_n} \textit{basis of the expression}$$
$$x \prod_{i=1}^{n} (1 + aJ(A_i)^{-1}).$$

For example, it allows us to express the $n-$th coefficient of Tr_{A_n} of a braid β as the 0-th coefficient of Tr_{A_n} of a "twisted braid" βFT_n^{-1}.

Markov trace

The generalized Markov traces can be computed as

- an explicit linear combination of characters of the Hecke algebra (Jones, Geck-Lambropoulou).
- This linear combination admits a uniform description as the Lusztig's Fourier transform of the Molien series of $S(V) \otimes \wedge(V)$, where V is the reflection representation (Gomi [2]).
- Webster and Williamson in [4] gave the first geometric interpretation of this uniform description.

In this project, we give a new simple formula for Markov traces in types B, D, similar to the one in type A, using the generalized Jucys-Murphy elements.

Markov trace in type B

Recall that y is a free parameter and $t_0^2 = \alpha_0 t_0 + 1$ where $\alpha_0 = v_0 - v_0^{-1}$, $t_i^2 = \alpha t_i + 1$ where $\alpha = v - v^{-1}$ and $i \neq 0$. Then the Markov trace in type B has the following expression.

Theorem

$$Tr_{B_n}^{v_0,v,y}(x) = \textit{coefficient near } 1 \textit{ in } \{t_w^{-1}\}_{w \in B_n} \textit{basis of the expression}$$

$$x \prod_{i=1}^{n} (1 + (y - \alpha_0) j(B_i)^{-1} + aJ(B_i)^{-1}).$$

Markov trace in type D

Corollary 1

$$Tr_{B_n}^{v_0=v,y=\alpha}(x) = \textit{coefficient near } 1 \textit{ in } \{t_w^{-1}\}_{w\in B_n} \textit{basis of the expression}$$
$$x\prod_{i=1}^{n}(1+aJ(B_i)^{-1}).$$

Corollary 2

$$Tr_{D_n}^{(k)}(x) = \textit{coefficient near } 1 \textit{ in } \{t_w^{-1}\}_{w\in D_n} \textit{basis of the coefficient near}$$
$$a^k \textit{ of the expression } x\prod_{i=1}^{n}(1+\sqrt{-a}(v-v^{-1})j(D_i)^{-1}+aJ(D_i)^{-1}).$$

Remarks

Note that in types B_n and D_{2n} $\widetilde{w}_0$ is central (while FT_n is always central). Let E_k denote the k-th elementary symmetric polynomial, then $Tr^{(k)}_{X_n}$ is given by the coefficient near 1 in $\{t_w^{-1}\}_{w \in W}$ basis of the following expressions:

Type A	$xE_k(J(A_1)^{-1}, \dots, J(A_n)^{-1})$
Type B	$xE_k(J(B_1)^{-1}, \dots, J(B_n)^{-1})$
Type D	$x \sum_{i=-k}^{k} (-1)^i v^{-2i} E_{k-i}(j(D_1)^{-1}, \dots, j(D_n)^{-1}) \times E_{k+i}(j(D_1)^{-1}, \dots, j(D_n)^{-1})$

For example, in type D for $k = 1$ the polynomial has the form $-v^2 E_2 + E_1^2 - v^{-2} E_2$. In particular, these elements in the table are central.

It turns out that $E_k(J(D_1), \dots, J(D_n))$ **is not** central. However, $E_k(1, J(D_2), \dots, J(D_n))$ **is** central.

References

[1] Meinolf Geck and Sofia Lambropoulou. *Markov traces and knot invariants related to Iwahori-Hecke algebras of type B*. 2004. arXiv: math/0405508 [math.RT]. URL: https://arxiv.org/abs/math/0405508.

[2] Yasushi Gomi. "The Markov traces and the Fourier transforms". In: *Journal of Algebra* 303.2 (2006). Computational Algebra, pp. 566–591. ISSN: 0021-8693. DOI: https://doi.org/10.1016/j.jalgebra.2005.09.034. URL: https://www.sciencedirect.com/science/article/pii/S0021869305005600.

[3] Vaughan FR Jones. "Hecke algebra representations of braid groups and link polynomials". In: *New Developments in the Theory of Knots*. World Scientific, 1987, pp. 20–73.

[4] Ben Webster and Geordie Williamson. "The geometry of Markov traces". In: *Duke Mathematical Journal* 160.2 (Nov. 2011). ISSN: 0012-7094. DOI: 10.1215/00127094-1444268. URL: