Ramsey Theory on Integers

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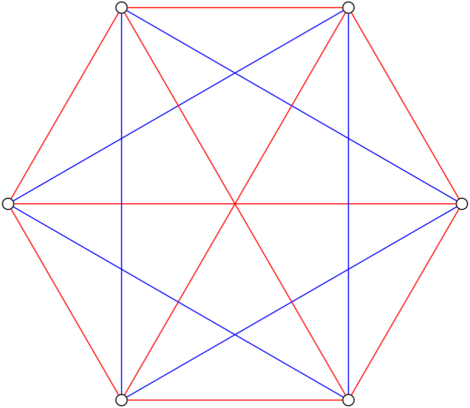
Section 1

Warm-up: Ramsey Theory on Graphs

Ramsey's Theorem for Graphs - Example



Ramsey's Theorem for Graphs

Theorem 1 (Ramsey's Theorem for two colors).

Let $k, l \geq 2$. There exists a least positive integer $R = R(k, l)$ such that every edge coloring of K_R, with the colors red and blue, admits either a red K_k subgraph or a blue K_l subgraph. $R(k, l)$ is called a Ramsey's number.

Ramsey's theorem for two colors can easily be generalized to $r \geq 3$ colors.

Ramsey's Numbers - Known Values

- $R(3,3) = 6$
- $R(3,4) = 9$
- $R(3,5) = 14$
- $R(3,6) = 18$
- $R(3,7) = 23$
- $R(3,8) = 28$
- $R(3,9) = 36$
- $R(4,4) = 18$
- $R(4,5) = 25$
- $R(5,5) = ??$

Section 2

Ramsey Theory on the Integers

Van der Waerden's Theorem

Theorem 2 (Van der Waerden's Theorem).

For all positive integers k and r, there exists a least positive integer $w(k;r)$ such that for every r-coloring of $\{1, \dots, w(k;r)\}$ there is a monochromatic arithmetic progression of length k.

r = 2, k = 3 (Coloring of $\{1, \dots, 9\}$)

i	1	2	3	4	5	6	7	8	9
$color(i)$	R	B	B	R	R	B	R	B	B

Monochromatic 3-term AP: (3, 6, 9) in Blue

Section 3

Proof of van der Waerden's Theorem

Notation

- We shall denote the set of integers by $\mathbb{Z}$, and the set of positive integers by $\mathbb{Z}^+$.
- An *interval* is a set of the form $\{a, a+1, ..., b\}$, where $a < b$ are integers. We will denote this interval more simply by $[a, b]$.
- An r-coloring of a set S is a function $\chi : S \to C$, where $|C| = r$.
- A coloring χ is monochromatic on a set S if χ is constant on S.

i	0	1	2	3	4	5	6
$\chi(i)$	0	1	2	0	1	2	0

Proposition (Translation)

Proposition 1 (Translation).

Let k, r, m, a, and b be positive integers. Every r-coloring of $[1, m]$ yields a monochromatic k-term arithmetic progression if and only if every r-coloring of

$$S = \{a, a + b, a + 2b...., a + (m - 1)b\}$$

yields a monochromatic arithmetic progression.

Refined Triples

Definition 1 (Refined Triples).

We say that a triple $(k, t; r)$ is refined if there exists a positive integer $m = m(k, t; r)$ such that for every r-coloring of $[1, m]$, there exist positive integers z, x_0, x_1, ..., x_t such that each of the sets

$$T_s = \left\{ b_s + \sum_{i=0}^{s-1} c_i x_i : c_i \in [1, k] \right\},$$

$0 \leq s \leq t$, is monochromatic, where

$$b_s = z + (k+1) \sum_{i=s}^{t} x_i.$$

Refined Triples - Example

Consider $k = 2$ and $t = 2$

$$T_0 = \{b_0\}$$
$$T_1 = \{b_1 + x_0, b_1 + 2x_0\}$$
$$T_2 = \{b_2 + x_0 + x_1, b_2 + 2x_0 + x_1, b_2 + x_0 + 2x_1, b_2 + 2x_0 + 2x_1\}$$

Refined Triples - Remark

If we only look at the elements in these sets where their coefficients are equal:

$$c_1 = c_2 = ... = c_{s-1} = j, \text{ for } j = 1, 2, 3, ..., k$$

we get an arithmetic progression of length k.

Example 1 ($k = 3$).

$$\begin{aligned} T_3 = \{ & b_3 + 1x_0 + 1x_1 + 1x_2, ..., \\ & b_3 + 2x_0 + 2x_1 + 2x_2, ..., \\ & b_3 + 3x_0 + 3x_1 + 3x_2\} \end{aligned}$$

Proof Outline

- Induction on k.
- **Lemma 1 ($w(k;r) \to (k,t;r)$).**
  Let $k > 1$. If $w(k;r)$ exists for all $r \geq 1$, then $(k,t;r)$ is refined for all $r, t \geq 1$.
- **Lemma 2 ($(k,t;r) \to w(k+1;r)$).**
  If $(k,t;r)$ is refined for all $r, t \geq 1$, then $w(k+1;r)$ exists for all $r \geq 1$.
- Lemma 1 + Lemma 2 give the induction step $k \to k+1$.

The Compactness Principle

Theorem 3 (The Compactness Principle).

If for every r-coloring of $\mathbb{Z}^+$ there is a monochromatic arithmetic progression, then there exists a least positive integer $n = n(r)$ such that for every r-coloring of $[1, n]$, there is a monochromatic arithmetic progression.

Lemma 2 Proof

Lemma 2 ($(k, t; r) \to w(k+1; r)$).
If $(k, t; r)$ is refined for all $r, t \geq 1$, then $w(k+1; r)$ exists for all $r \geq 1$.

Proof.
Let r be given and let χ be any r-coloring of $\mathbb{Z}^+$.

By assumption, $(k, t; r)$ is refined for all $r, t \geq 1$. In particular, $r = t$.

By definition of refined triples, there exist z, x_0, ..., x_r such that each of the sets T_0, T_1, ..., T_r is monochromatic under χ. By the pigeonhole principle, two of these sets must be the same color. Let T_v and T_w, $v < w$ be such sets.

Lemma 2 Proof

$$T_v = \left\{ z + (k+1)\sum_{i=v}^{r} x_i + \sum_{i=0}^{v-1} c_i x_i : c_i \in [1,k] \right\}$$

and

$$T_w = \left\{ z + (k+1)\sum_{i=w}^{r} x_i + \sum_{i=0}^{w-1} c_i x_i : c_i \in [1,k] \right\}$$

Letting $a = z + \sum_{i=0}^{v-1} x_i + (k+1)\sum_{i=w}^{r} x_i$, we rewrite these as

$$T_v = \left\{ a + (k+1)\sum_{i=v}^{w-1} x_i + \sum_{i=0}^{v-1} (c_i - 1) x_i : c_i \in [1,k] \right\}$$

$$T_w = \left\{ a - \sum_{i=0}^{v-1} x_i + \sum_{i=0}^{w-1} c_i x_i : c_i \in [1,k] \right\}$$

Lemma 2 Proof

Taking $c_0 = c_1 = ... = c_{v-1} = 1$ in T_w we have

$$T'_w = \left\{ a + \sum_{i=v}^{w-1} c_i x_i : c_i \in [1, k] \right\} \subseteq T_w$$

Letting $d = \sum_{i=v}^{w-1} x_i$, we have $a + (k+1)d \in T_v$.

Hence, we have found a monochromatic arithmetic progression of length $k+1$, thereby proving the existence of $w(k+1; r)$.

Lemma 1 Proof

Lemma 1 ($w(k; r) \to (k, t; r)$).
Let $k > 1$. If $w(k; r)$ exists for all $r \geq 1$, then $(k, t; r)$ is refined for all $r, t \geq 1$.

The proof is by induction on t, starting with $t = 1$.

Lemma 1 Proof - Base Case

$t = 1$

To prove that $(k, 1; r)$ is refined, we first show that we may take $m = m(k, t; r)$ to be $3w(k; r) + k + 1$.

Let χ be an arbitrary r-coloring of $[1, m] = [1, 3w(k; r) + k + 1]$.

Since we are assuming that $w(k; r)$ exists, applying translation, the interval $[w(k; r) + k + 2, 2w(k; r) + k + 1]$ must admit a monochromatic k-term arithmetic progression $S = \{a + d, a + 2d, ..., a + kd\}$.

Lemma 1 Proof - Base Case

Using the notation of Refined Triples, let

$$z = a - (k+1), x_0 = d, x_1 = 1$$

This gives:

$$T_0 = \{a + (k+1)d\}$$
$$T_1 = S$$

which are both contained in $[1, m]$ and are both monochromatic.

Thereby proving that $(k, 1; r)$ is a refined triple.

Lemma 1 Proof - Induction Step

Let $t \geq 1$ and assume that $(k, t; r)$ is refined.

We will show that $(k, t + 1; r)$ is refined.

Derived Coloring

Let r, m, $n \geq 1$. Let γ be an r-coloring of $[1, n+m]$. Define $\chi_{\gamma,m}$ to be the r^m-coloring of $[1, n]$ as follows: for $j \in [1, n]$, let $\chi_{\gamma,m}(j)$ be the m-tuple $(\gamma(j+1), \gamma(j+2), \ldots, \gamma(j+m))$. We call $\chi_{\gamma,m}$ a coloring *derived* from γ.

i	1	2	3	4	5	6
$\gamma(i)$	1	0	2	1	1	0
$\chi_{\gamma,m}$	(0, 2, 1)	(2, 1, 1)	(1, 1, 0)	(1, 0, 2)	...	...

Lemma 1 Proof - Induction Step

Suppose $m = m(k, t; r)$ exists, and let $n = 2w(k; r^m)$.

We claim that we may take $m(k, t + 1; r) = n + m$.

Let γ be an r-coloring of $[1, n + m]$. Let $\chi = \chi_{\gamma,m}$ be the r^m-coloring of $[1, n]$ *derived* from γ.

By the definition of n, and since $w(k; r^m)$ exists, there must be an arithmetic progression:

$$\{a + d, a + 2d, ..., a + (k + 1)d\} \subseteq [1, n]$$

with the first k terms monochromatic under χ.

Lemma 1 Proof - Induction Step

By the definition of χ, the k intervals $I_j = [a + jd + 1, a + jd + m], 1 \leq j \leq k$, have identical colorings under γ.

Since $(k, t; r)$ is refined, there exist z, x_0, x_1, ..., x_t that the T_i's are monochromatic under γ. Therefore, since I_j have identical colorings, each I_j contains the monochromatic sets:

$$\begin{aligned} S_s(j) &= T_s + (a + jd) \\ &= \{y + a + jd : y \in T_s\} \\ &= \left\{(b_s + a + jd) + \sum_{i=0}^{s-1} c_i x_i : c_i \in [1, k]\right\} \end{aligned}$$

for $s = 0, 1, ..., t$

Lemma 1 Proof - Induction Step

Furthermore, since the intervals have the same coloring under γ, $S_s(u)$ and $S_s(v)$ must have the same coloring under γ for $1 \leq u, v \leq k$.

Hence, by construction, the set

$$\begin{aligned} Q_s &= \bigcup_{v=1}^{k} S_s(v) \\ &= \left\{ (b_s + a) + \sum_{i=0}^{s-1} c_i x_i + jd : j, c_i \in [1, k] \right\} \end{aligned}$$

is monochromatic under γ for each $s = 0, 1, ..., t$.

Lemma 1 Proof - Induction Step

Now we define sets $T'_0, ..., T'_{t+1}$ that satisfy the definition of refined triples.

Let

$$z' = z + a$$
$$x'_0 = d$$
$$x'_i = x_{i-1} \text{ for } i = 1, 2, ..., t + 1$$

Check: $T'_{s+1} = Q_s$ for $s = 0, 1, ..., t$.

Remains: T'_0 is always monochromatic.

Thus we have satisfied the conditions required to prove that $(k, t + 1; r)$ is refined, thereby proving the lemma.

Outro

Any Questions?