Turan's Theorem

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25/06/2025

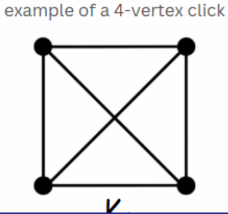
Central Question

Central Question for Turan's Theorem

What is the maximal number of edges in a simple graph with n vertices that **does not** contain a fixed subgraph **H**?

Definitions

- $G = (V, E)$: a graph with $n = |V|$ vertices
- $e(G)$: the number of edges of G
- $ex(n, F) = \max\{e(G) : |V(G)| = n, F \text{ not a subgraph of } G\}$.
- $ex(n, F)$: the extremal function forbidding subgraph F



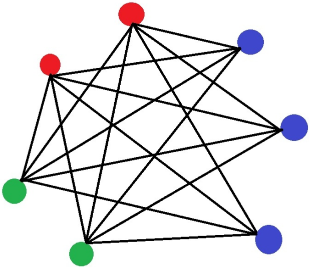
Turan's Graph

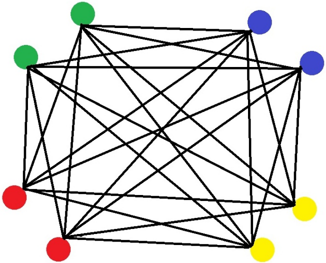
Theorem

Turan's theorem states that max amount of edges graph
$ex(n, K_{r+1}) \leq (1 - \frac{1}{r})\frac{n^2}{2}$

Theorem

The maximum number of edges in a graph $ex(n, K_{r+1})$ is achieved if and only if graph is Turan's graph. Turan's graph is achieved by having r-partite graph with vertices in graph being split as evenly as possible





Proof of Turan's Theorem

Theorem

The maximum number of edges a graph without $r+1$ clique can have is achieved by Turan's graph, which is r-partite graph with vertices in graph being split as evenly as possible

Keypoints

- We have a graph $G(V, E)$. Induction is going on r.
- Base case r $=$ 1 is obvious its just 1 vertex.
- Assume that r $>$ 1, meaning $ex(n, K_r) = e(T_{n,r-1})$ for any n
- Vertex v is vertex with most edges in graph G
- The set A is a set of all vertices connected to v
- The set B is a set of all vertices not connected to v

Proof of Turan's Theorem

Keypoints

- If graph G is K_{r+1}-free then A is K_r-free
- $e(A) \leq ex(|A|, K_r)$
- Maximum amount between A and B and within B is $|A| \cdot |B|$

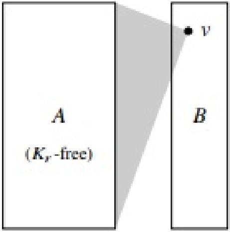


Figure: Visual example of A

Turan Density

Density Definition

$\pi(H) := \lim_{n\to\infty} \frac{ex(n,H)}{\binom{n}{2}}$

$\pi(H)$ for K_{r+1}

$\pi(K_{r+1}) = 1 - \frac{1}{r}$

Erdős–Stone–Simonovits theorem

$\pi(H) = 1 - \frac{1}{\chi(H)-1}$

Zarankiewich Problem

Problem statement

Determine $ex(n, K_{s,t})$, the maximum number of edges in an n-vertex $K_{s,t}$ -free graph.

Kővári–Sós–Turán theorem – "KST theorem"

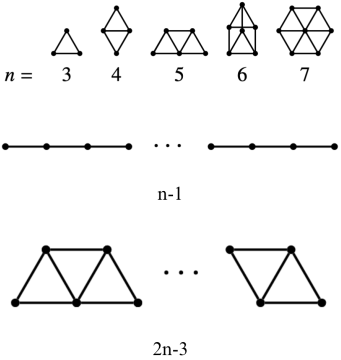
For positive integers $s \leq t$, there exists some constant $C = C(s, t)$, such that, for all n, $ex(n, K_{s,t}) \leq Cn^{2-1/s}$

Consequence

For every bipartite graph H, there exists some constant $c > 0$ such that $ex(n, H) = O_H(n^{2-c})$.

Famous problem was posed by Erdős (1946)

What is the maximum number of unit distances formed by a set of n points in $\mathbb{R}^2$?



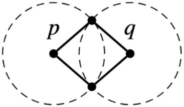
Geometric Applications: Upper bound

Upper bound

Every set of n points in $\mathbb{R}^2$ has $O(n^{3/2})$ unit distances.

Proof

Every unit distance graph is $K_{2,3}$-free. For every pair of distinct points, there are at most two other points that are at a unit distance from both points.
The number of edges is at most $ex(n, K_{2,3}) = O(n^{3/2})$ by KST theorem.



Forbidding a subgraph

Open Problems: Turan's Theorem for Hypergraphs

Keypoints

- A k-uniform hypergraph: each edge connects k vertices (not just 2)
- Extremal question:
- $ex(n, K_{r+1}^{(k)})$ = max number of edges avoiding $K_{r+1}^{(k)}$
- Known for graphs ($k = 2$), but unknown for most $k \geq 3$
- Example: open problem — $ex(n, K_4^{(3)})$
- Area of active research: flag algebras, probabilistic methods, design theory