# Bounds on the number of edges in visibility hypergraphs

Alexander Wang

June 19, 2026

**Abstract**

We introduce semi-bar, arc, semi-arc, and rectangle $k$-visibility hypergraphs, which are generalizations of the corresponding $k$-visibility graphs. We prove optimal bounds on the maximum number of edges in terms of $k$ and $n$, the number of vertices, in four classes of visibility hypergraphs: bar, semi-bar, arc, and semi-arc $k$-visibility hypergraph. The results on bar $k$-visibility hypergraphs solve an open problem from (Geneson and Shen, 2014). Additionally, we prove the maximum number of edges in any arc $k$-visibility graph when $n \geq 7k + 6$, solving an open problem from (Sawhney and Weed, 2015). We also prove lower and upper bounds on the number of edges in rectangle $k$-visibility hypergraphs. The research is fundamental for both theoretical understanding of visibility graphs and practical applications in computational geometry, robot path planning, and VLSI layout design.

## 1 Introduction

A *visibility graph* is a graph in which each vertex is represented as a region so that there is an edge between two vertices if and only if the two regions are visible to each other. We investigate bounds on the maximum number of edges for multiple different types of visibility hypergraphs, including bar, semi-bar, arc, semi-arc, and rectangle $k$-visibility hypergraphs.

Visibility graphs and hypergraphs are widely used in computational geometry, robot path planning, and very-large-scale integration (VLSI) layout design because they capture spatial relations in a combinatorial structure and provide geometric representations of graphs for better readability. For example, VLSI chips can be modeled as a bar, arc or rectangle visibility graph or hypergraph, where the macrocells represent the vertices of the graph and the edges are connections between the macrocells [12]. Studying the visibility graph, particularly the maximum number of edges, can significantly reduce the size, number of layers, and complexity of the chip, therefore leading to more compact and efficient design. This has become increasingly significant as modern mobile and AI chips demand reduced size and improved power efficiency. Another application of the visibility graph is path planning for a robot to find the shortest path in a environment with obstacles.

The maximum edge bounds in this research are fundamental for both theoretical understanding and practical applications of visibility graphs and hypergraphs. It aids the structural understanding of the graphs, as the maximum number of edges tells us how dense these graphs can get. Knowing these edge bounds also helps us decide whether a given graph admits a visibility representation. It also gives bounds on the theoretical worst case complexity for many graph algorithms, for example finding shortest paths and thickness in these visibility graphs. Edge bounds also give us practical limits in applications like VLSI design, network and information visualization.

### 1.1 Bar and semi-bar $k$-visibility graphs

Bar visibility graphs were originally introduced by Duchet et al. [5] and Schlag et al. [13] in connection to VLSI circuit design.

**Definition 1.1.** A *bar visibility graph* is a visibility graph where each vertex is represented in the plane as a horizontal line segment called a bar, and two bars are visible to each other if and only if there exists a vertical line segment that intersects the two bars and no other bars. A *bar visibility representation* is the representation of all of the bars corresponding to vertices. An edge can be drawn on the bar visibility representation as some vertical line segment intersecting exactly two bars.

A bar visibility graph is shown in Figure 1. Bar visibility graphs were used by Fulek [7] to prove an upper bound on the extremal function of a certain 0–1 matrix. A matrix $A$ contains $B$ if it is possible to delete some rows and columns of $A$ and replace some 1s in $A$ with 0s to obtain $B$. He showed that the number of 1s in a 0–1 matrix with $n$ rows and $n$ columns that does not contain the matrix $\begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix}$ is at most $7n-13$ by constructing a bar visibility graph using matrix entries.

Bar $k$-visibility graphs are generalizations of bar visibility graphs.

**Definition 1.2.** A *bar $k$-visibility graph* is a visibility graph in which an edge connects two vertices if and only if there is a vertical segment between the two bars that intersects at most $k$ other bars.

In particular, a bar visibility graph is a bar 0-visibility graph. Dean et al. [2] proved that the maximum number of edges in a bar 1-visibility graph with $n$ vertices is $6n-20$ and that the maximum number of edges for a bar $k$-visibility graph with $n \geq 4k+4$ vertices is between $(k+1)(3n-4k-6)$ and $(k+1)\left(3n-\frac{7}{2}k-5\right)-1$. Hartke et al. [9] refined the edge counting technique in the upper bound and proved that the exact maximum is equal to $(k+1)(3n-4k-6)$.

**Definition 1.3.** A *semi-bar $k$-visibility graph* is a bar $k$-visibility graph where all of the bars in the representation have their left endpoint at the same $x$-coordinate.

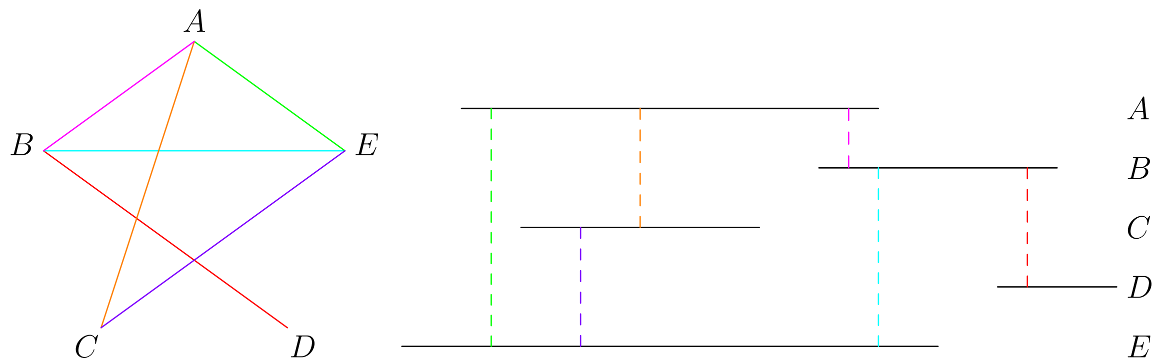


Figure 1: Bar visibility graph (left) and its corresponding bar visibility representation (right), with corresponding edges colored


Figure 2: Arc visibility graph (left) and its corresponding arc visibility representation (right), with corresponding edges colored

Felsner and Massow [6] proved that the maximum number of edges in a semi-bar $k$-visibility graph on $n$ vertices is $(k+1)(2n-2k-3)$ when $n \geq 2k+3$ and $\binom{n}{2}$ when $n \leq 2k+2$.

### 1.2 Arc and semi-arc $k$-visibility graphs

Arc visibility graphs were first introduced by Hutchinson [10].

**Definition 1.4.** An *arc $k$-visibility graph* is a visibility graph where each vertex is represented as an arc centered at the origin, and each edge can be represented as a line segment collinear with the origin, possibly passing through the origin, that intersects the two arcs and at most $k$ other arcs. An *arc visibility graph* is an arc 0-visibility graph.

An arc visibility graph is shown in Figure 2. Babbitt et al. [1] proved that the maximum number of edges in an arc $k$-visibility graph with $n$ vertices is upper bounded by $(k+1)(3n-k-2)$ for $n \geq 4k+5$. Later, Sawhney and Weed [14] improved the upper bound to $(k+1)\left(3n-\frac{3k+6}{2}\right)$. This bound was known to be optimal for $k=0$, which gives the maximum number of edges in an arc visibility graph on $n$ vertices is $3n-3$.

**Definition 1.5.** A *semi-arc $k$-visibility graph* is an arc $k$-visibility graph where every arc extends counterclockwise from the same angular position.

Babbitt et al. [1] proved that a semi-arc $k$-visibility graph with $n \geq 3k+3$ vertices has at most $(k+1)\left(2n-\frac{k+2}{2}\right)$ edges and gave a construction with $(k+1)\left(2n-\frac{3k+6}{2}\right)$ edges. Sawhney and Weed [14] proved that their upper bound was the true maximum by constructing a semi-arc $k$-visibility graph on $n \geq 3k+3$ vertices with $(k+1)\left(2n-\frac{k+2}{2}\right)$ edges.

### 1.3 Rectangle $k$-visibility graphs

**Definition 1.6.** A *rectangle visibility graph* is a visibility graph where each vertex is represented as an axis-aligned rectangle, and two vertices are connected by an edge if there exists a horizontal or vertical line segment intersecting exactly the two rectangles corresponding to the vertices.

Dean and Hutchinson [3] considered representations of bipartite graphs as rectangle visibility graphs, characterizing all complete bipartite graphs which are rectangle visibility graphs and showing that any bipartite rectangle-visibility graph on $n \geq 4$ vertices has at most $4n-12$ edges. Hutchinson et al. [11] proved that the maximum number of edges in a rectangle visibility graph on $n \geq 8$ vertices is $6n-20$ and provided a construction that has $6n-20$ vertices.

**Definition 1.7.** A *rectangle $k$-visibility graph* is a graph where each vertex is drawn as an axis-aligned rectangle and each edge is a vertical or horizontal segment connecting two rectangles which intersects at most $k$ other rectangles.

Slettnes first defined rectangle $k$-visibility graphs in [15]. For fixed $k$, the maximum number of edges in a rectangle $k$-visibility graph with $n$ vertices is linear in $n$ (bounded above by $(6k+6)n$), but the coefficient of $n$ is unknown.

### 1.4 Bar visibility hypergraphs

Another generalization of bar visibility graphs are bar $k$-visibility hypergraphs.

**Definition 1.8.** A *bar $k$-visibility hypergraph* is a hypergraph where the vertices are represented as bars, and each edge contains exactly $k+2$ vertices corresponding to all bars that intersect some vertical segment.

Bar $k$-visibility hypergraphs were first introduced by Geneson and Shen [8] to generalize Fulek's argument [7] to a larger class of matrices. They proved that a bar $k$-visibility hypergraph with $n$ vertices has at most $(2k+3)n$ edges and used this bound to prove a linear upper bound on the number of 1s in a 0–1 matrix with $n$ rows and $n$ columns that does not contain the matrix $\begin{pmatrix} 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}$.

### 1.5 Results

We propose the following definitions for four new types of visibility hypergraphs: semi-bar, arc, semi-arc, and rectangle $k$-visibility hypergraphs.

**Definition 1.9.** A *semi-bar $k$-visibility hypergraph* is a bar $k$-visibility hypergraph where all bars have the same $x$-coordinate at the left endpoint.

**Definition 1.10.** An *arc $k$-visibility hypergraph* has vertices which are arcs centered around the origin, and each edge contains $k+2$ vertices which correspond to a line segment collinear with the origin intersecting exactly $k+2$ distinct arcs.

**Definition 1.11.** A *semi-arc $k$-visibility hypergraph* is an arc $k$-visibility hypergraph where all arcs begin at the same angular position.

**Definition 1.12.** A *rectangle $k$-visibility hypergraph* has vertices which are axis-aligned rectangles and each edge contains exactly $k+2$ rectangles that intersect a horizontal or vertical line segment.

When $k=0$, these definitions reduce to semi-bar, arc, semi-arc, and rectangle visibility graphs.

In Section 2, we obtain optimal bounds on the maximum number of edges in a bar $k$-visibility hypergraph. We improve on Geneson and Shen's upper bound [8] by proving that the maximum number of edges in a bar $k$-visibility hypergraph on $n$ vertices is equal to $(2k+3)n - 3k^2 - 8k - 6$ when $n \geq 2k+3$, which strengthens their bound of the extremal function in their paper. Furthermore, we prove that the maximum number of edges in a semi-bar $k$-visibility hypergraph on $n$ vertices is exactly $\frac{(k+2)(2n-3k-3)}{2}$ for $n \geq 2k+4$ and $\frac{(n-1-k)(n-k)}{2}$ for $k+2 \leq n \leq 2k+3$.

In Section 3, we provide bounds for the maximum number of edges in semi-arc $k$-visibility hypergraphs and arc $k$-visibility graphs. We prove that the maximum number of edges in a semi-arc $k$-visibility hypergraph on $n \geq k+1$ vertices is at most $n(k+2) - \frac{(k+1)(k+2)}{2}$, and this upper bound is optimal when $n \geq 3k+5$. We also prove that the maximum number of edges in an arc $k$-visibility hypergraph on $n \geq \max(6, 2k+5)$ vertices is $n(2k+3) - (k+1)(k+3)$.

In Section 4, we prove that Sawhney and Weed's upper bound of $3n(k+1) - \frac{3(k+1)(k+2)}{2}$ [14] for the number of edges in an arc $k$-visibility graphs with $n$ vertices is optimal when $n \geq 7k+6$ by providing a construction with $3n(k+1) - \frac{3(k+1)(k+2)}{2}$ edges.

In Section 5, we provide bounds on the maximum number of edges in a rectangle $k$-visibility hypergraph. We use a construction based on the construction given by Hutchinson et al. in [11].

## 2 Bar and semi-bar $k$-visibility hypergraphs

We find the maximum number of edges in bar and semi-bar $k$-visibility hypergraphs with a fixed number of vertices $n$. Geneson and Shen [8] proved that the number of edges in a bar $k$-visibility hypergraph with $n$ vertices is at most $(2k+3)n$. We improve this bound, and provide optimal bound for bar and semi-bar $k$-visibility hypergraphs.

Throughout the proofs in this section, we assume that no two bar endpoints share the same $x$-coordinate because if this happens, we can change the locations of the endpoints slightly while not decreasing the number of edges.

**Theorem 2.1.** *The maximum number of edges in a bar $k$-visibility hypergraph with $n$ vertices is $(2k+3)n - 3k^2 - 8k - 6$ if $n \geq 2k+3$.*

*Proof.* We will start with a construction achieving this bound. Draw the horizontal bars $A_i$ from $(2i, i)$ to $(2n-1-2i, i)$ for $0 \leq i \leq k$, bars $B_i$ from $(2i+1, n-1-i)$ to $(2n-2i-2, n-1-i)$ for $0 \leq i \leq k$, bars $C_i$ from $(2k+2+2i, k+1+i)$ to $(2k+5+2i, k+1+i)$ for $0 \leq i \leq n-2k-4$, and bar $C_{n-2k-3}$ from $(2n-2s-4, n-k-2)$ to $(2n, n-k-2)$. The case $k=2$ and $n=10$ is shown in Figure 3.

We will show that there are $(2k+3)n - 3k^2 - 8k - 6$ edges in this hypergraph. Each edge intersects at most two of the $C_i$ because no three of the $C_i$ share the same $y$-coordinate.

If an edge intersects two of the $C_i$, then they must be $C_i$ and $C_{i+1}$ for some $0 \leq i \leq n-2k-4$. Then, an edge can intersect the bars $A_j$ to $A_k$, $C_i$, $C_{i+1}$, and $B_k$ to $B_{k-j+2}$

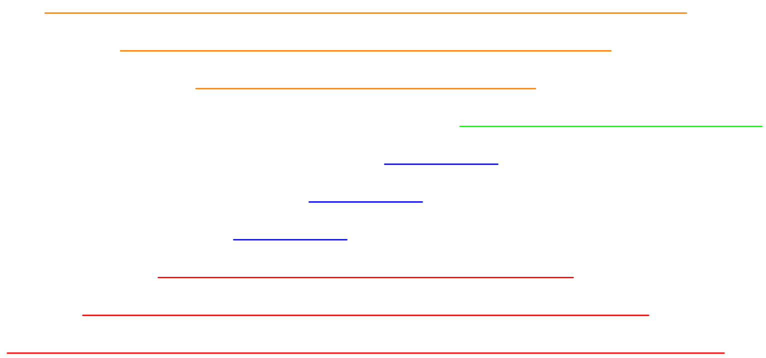

Figure 3: Construction for $n = 10$ and $k = 2$

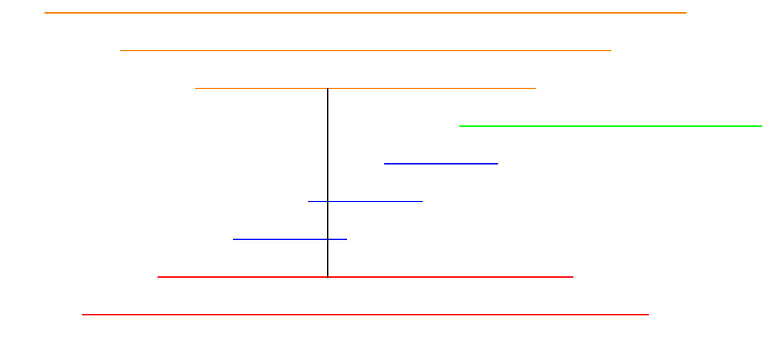

Figure 4: An edge in the first case of Theorem 2

for $1 \leq j \leq k+1$. In this case, there are $(n-2k-3)(k+1)$ edges. This case is shown in Figure 4.

If an edge intersects exactly one of the $C_i$ for some $0 \leq i \leq n-2k-4$, then an edge can intersect the bars $A_j$ to $A_k$, $C_i$, and $B_k$ to $B_{k-j+1}$ for $0 \leq j \leq k+1$. In this case, there are $(n-2k-3)(k+2)$ edges. This case is shown in Figure 5.

If an edge intersects $C_{n-2k-3}$ but not any of the other $C_i$, then the $x$-coordinate of the edge must be between $2n-2k-4$ and $2n$. For all $2n-2k-3 \leq i \leq 2n-1$, there are $2n-i$ bars containing a point with $x$-coordinate between $i$ and $i+1$. This gives

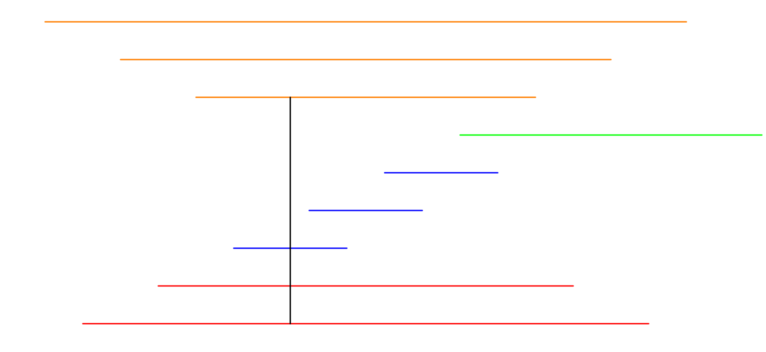

Figure 5: An edge in the second case of Theorem 2

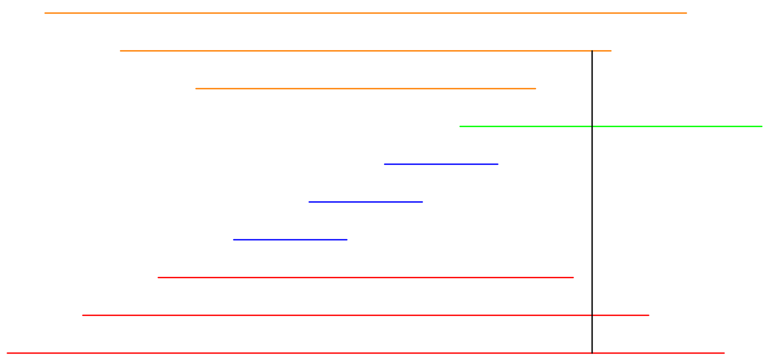

Figure 6: An edge in the third case of Theorem 2

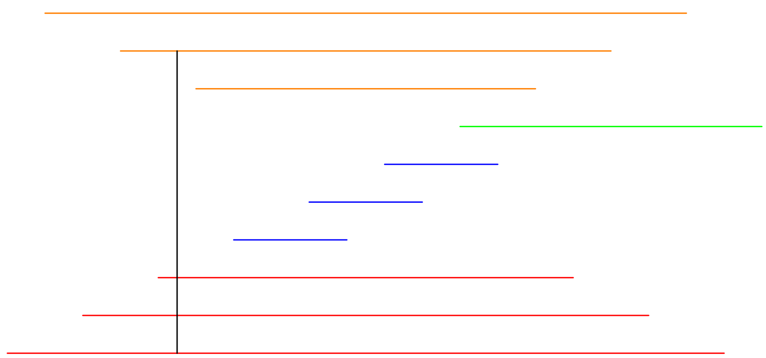

Figure 7: An edge in the fourth case of Theorem 2

$\max(2n-i-k-1,0)$ possible edges. Each of these possible edges contains $C_{n-2k-3}$. Each one appears for exactly one $i$ except for the edge containing bars $A_0$ to $A_s$ and $C_{n-2k-3}$, which is counted twice for $i = 2n-2k-4$ and $i = 2n-2k-3$. In this case, there are $(1+2+\cdots+(k+2))-1 = \frac{(k+2)(k+3)}{2} - 1 = \frac{k^2+5k+4}{2}$ edges. This case is shown in Figure 6.

If an edge does not intersect any of the $C_i$, then the $x$-coordinate of the edge must be between 0 and $2k+1$. For all $0 \le i \le 2k$, there are $i+1$ bars containing a point with $x$-coordinate between $i$ and $i+1$. This gives $\max(i-k,0)$ possible edges. Each of these edges are distinct over all $i$ since for any $i$, each edge with $x$ coordinate between $i$ and $i+1$ must have $A_{\lfloor \frac{i}{2} \rfloor}$ and $B_{\lfloor \frac{i-1}{2} \rfloor}$ consecutive. Therefore, the number of edges in this case is $1+2+\cdots+(k+1) = \frac{(k+1)(k+2)}{2} = \frac{k^2+3k+2}{2}$. This case is shown in Figure 7.

The total number of edges is

$$(n-2k-3)(k+1)+(n-2k-3)(k+2)+\frac{k^2+5k+4}{2}+\frac{k^2+3k+2}{2} = n(2k+3)-3k^2-8k-6.$$

Now, we will show that no bar $k$-visibility hypergraph can have more than $n(2k+3)-3k^2-8k-6$ edges.

Shift each edge to the leftmost possible $x$-coordinate, as shown in Figure 8. Then, each edge is at the left edge or right edge of exactly one bar. Order the bars from 1 to $n$. Consider the $i$th bar.

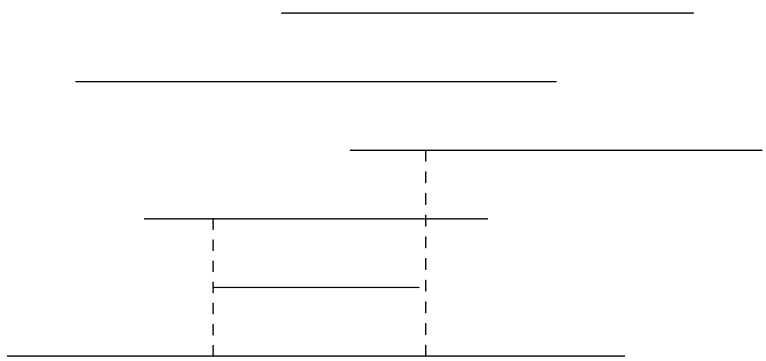

Figure 8: One edge on the left side and the right side of a bar

If an edge is on the left side of the bar, the edge must intersect $k+1$ other bars next to the $i$th bar. As there are at most $i-1$ other bars containing a point with the same $x$-coordinate as the left side of the bar, the number of edges is at most $\min(k+2, \max(i-k-1, 0))$.

If an edge is on the right side of the bar, the edge must intersect $k+2$ other bars next to the $i$th bar such that the $i$th bar is between some two of the other bars. At most $n-i$ other bars have a point with the same $x$-coordinate as the right side of the $i$th bar. Therefore, the number of edges is at most $\min(k+1, \max(n-i-k-1, 0))$.

The number of edges is upper bounded by

$$\begin{aligned}
&\sum_{i=1}^{n} \min(k+2, \max(i-k-1, 0)) + \sum_{i=1}^{n} \min(k+1, \max(n-i-k-1, 0)) \\
&= \sum_{i=k+1}^{2k+2} (i-k-1) + \sum_{i=2k+3}^{n} (k+2) + \sum_{i=1}^{n-2s-2} (k+1) + \sum_{i=n-2k-1}^{n-k-1} (n-i-k-1) \\
&= \frac{(k+1)(k+2)}{2} + (n-2k-2)(k+2) + (n-2k-2)(k+1) + \frac{k(k+1)}{2} \\
&= n(2k+3) - 3k^2 - 8k - 5.
\end{aligned}$$

If the number of edges is equal to $n(2k+3) - 3k^2 - 8k - 5$, then equality must hold for every term. In particular, the left side of the $i$th bar must have $k+2$ edges for $2k+3 \leq i \leq n$. This is only possible if the $i$th bar has at least $k+1$ bars below it and $k+1$ bars above it. Therefore, the top $k+1$ bars and the bottom $k+1$ bars must have the $2k+2$ leftmost left endpoints. This means that the edge consisting of the top $k+1$ bars and the bottom $k+1$ bars is counted once on the left endpoint of the $2k+2$th bar. Similarly, the right side of the $i$th bar must have $k+1$ edges for $1 \leq i \leq n-2k-2$, which means the $i$th bar must have at least $k+1$ bars below it and $k+1$ bars above it. This means the top $k+1$ bars and the bottom $k+1$ bars must have the $2k+2$ rightmost right endpoints, so the edge consisting of the top $k+1$ bars and the bottom $k+1$ bars is also counted once on the right endpoint of the $n-2k-2$th bar, which is a contradiction as this edge is counted twice, so the number of edges is less than $n(2k+3) - 3k^2 - 8k - 5$.

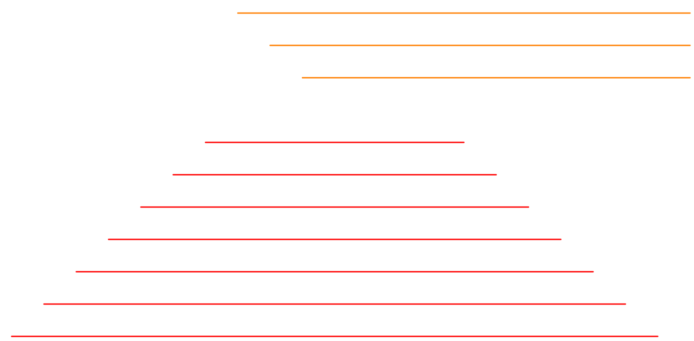

Figure 9: Construction for $n = 10$ and $k = 6$

Therefore, the maximum number of edges is $n(2k+3) - 3k^2 - 8k - 6$. □

We now prove optimal bounds in the case $n \leq 2k+2$. Since each edge must intersect $k+2$ bars, there are 0 edges when $n \leq k+1$. Thus, we are left with the case $k+2 \leq n \leq 2k+2$.

**Theorem 2.2.** *The maximum number of edges in a bar $k$-visibility hypergraph with $n$ vertices is $(n-k-1)^2$ if $k+2 \leq n \leq 2k+2$.*

*Proof.* The proof of the upper bound is similar to the upper bound proof of Theorem 2. The number of edges on the left side of the $i$th bar is at most $\max(i-k-1, 0)$, and the number of edges on the right side of the $i$th bar is $\max(n-i-k-1, 0)$. Therefore, the total number of edges is at most

$$\sum_{i=1}^{n} \max(i-k-1, 0) + \sum_{i=1}^{n} \max(n-i-k-1, 0) = (n-k-1)^2.$$

Now, we describe the construction which achieves this bound. Construct bars $A_i$ from $(i, i)$ to $(2n-i, i)$ for $0 \leq i \leq k$, and construct bars $B_i$ from $(k+1+i, n-i)$ to $(2n+1, n-i)$ for $0 \leq i \leq n-k-2$. The case $n = 10$ and $k = 6$ is shown in Figure 9. For each $0 \leq i \leq j < n-k-2$, there is an edge that passes through $B_i$, $B_{i+1}$, $\ldots$, $B_j$, $A_k$, $A_{k-1}$, $\ldots$, $A_{k-(j-i)}$. There are $\frac{(n-k-2)(n-k-1)}{2}$ such edges. Additionally, for each $0 \leq i \leq j \leq k$ satisfying $j - i \geq 2k+2-n$, there exists an edge passing through $A_i$, $A_{i+1}$, $\ldots$, $A_j$, $B_{n-k-2}$, $\ldots$, $B_{n-2k-2+(j-i)}$. There are $\frac{(n-k-1)(n-k)}{2}$ such edges. All of these edges are distinct and there are no other edges, so this construction has a total of $(n-k-1)^2$ edges.

□

**Theorem 2.3.** *The maximum number of edges in a semi-bar $k$-visibility hypergraph with $n$ vertices is*

- $0$ *when* $n \leq k$,
- $\frac{(n-1-k)(n-k)}{2}$ *when* $k+1 \leq n \leq 2k+3$, *and*
- $\frac{(k+2)(2n-3k-3)}{2}$ *when* $n \geq 2k+4$.

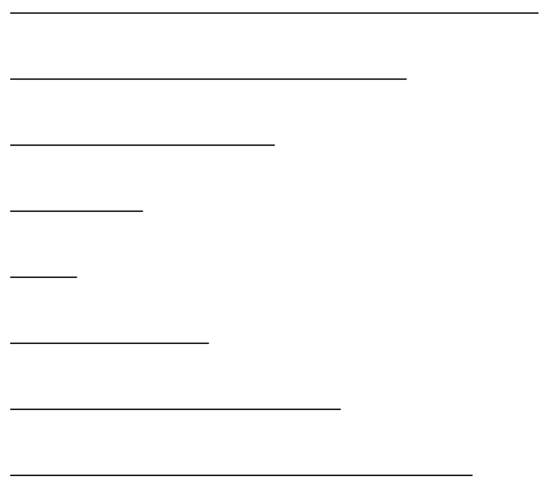

Figure 10: Construction for $n = 8$

*Proof.* Shift each edge to the rightmost possible position. Then, each edge is moved to the right endpoint of some bar. Order the bars from 1 to $n$ based on the $x$-coordinate of the right endpoint. Consider the edges on the right endpoint of the $i$th bar. Each edge must intersect $k+1$ other bars which are all consecutive to the $i$th bar. There are at most $n-i$ such bars, so the number of edges is at most $\min(k+2, \max(n-i-k, 0))$.

If $n \leq 2k+3$, then the expression is always equal to $\max(n-i-k, 0)$, so the total number of edges is upper bounded by 0 when $n \leq k$ and $\frac{(n-1-k)(n-k)}{2}$ when $k+1 \leq n \leq 2k+3$. If $n \geq 2k+4$, the total number of edges is at most

$$\begin{aligned}\sum_{i=1}^{n} \min(k+2, \max(n-i-k, 0)) &= \sum_{i=1}^{n-2k-3} (k+2) + \sum_{i=n-2k-2}^{n-k} (n-i-k) \\ &= (n-2k-3)(k+2) + \frac{(k+2)(k+3)}{2} \\ &= \frac{(k+2)(2n-3k-3)}{2}.\end{aligned}$$

Now, we provide a construction that achieves this upper bound. For $1 \leq i \leq n$, draw a bar from $(0, (-1)^i \lfloor \frac{i}{2} \rfloor)$ to $(i, (-1)^i \lfloor \frac{i}{2} \rfloor)$. The construction for $n = 8$ is shown in Figure 10.

Consider the edges with $x$-coordinate $i$ for $1 \leq i \leq n$. If $1 \leq i \leq n-2k-2$, there are $k+2$ edges. Otherwise, if $n-2k-1 \leq i \leq n$, then there are at most $k+1$ bars above and below the bar ending at this $x$-coordinate, and there are $n+1-i$ total bars at this $x$-coordinate, so the total number of edges is $\max(n-i-k, 0)$. All of these edges are distinct, so this construction achieves the upper bound of

$$\sum_{i=1}^{n} \min(k+2, \max(n-i-k, 0)) = \begin{cases} 0 & n \leq k \\ \frac{(n-1-k)(n-k)}{2} & k+1 \leq n \leq 2k+3 \\ \frac{(k+2)(2n-3k-3)}{2} & n \geq 2k+4 \end{cases}.$$

□

## 3 Arc and semi-arc $k$-visibility hypergraphs

In this section, we prove bounds for the maximum number of edges in arc and semi-arc $k$-visibility hypergraphs. Our bounds for semi-arc $k$-visibility graphs are optimal when $n \geq 3k+5$, and our bounds for arc $k$-visibility graphs are optimal when $n \geq 2k+5$.

In this section and the next section, we assume that no two arc endpoints lie on the same line passing through the origin because if this happens, we can shift the endpoints slightly and not decrease the number of edges. Additionally, we describe the endpoints of the arcs using polar coordinates. If any two arcs have the same radius, we can slightly change the radius of one of the arcs while keeping the same visibilities, so we can assume that all arc radii are distinct. Furthermore, we can make the radii 1, 2, …, $n$, where $n$ is the number of arcs.

For a semi-arc $k$-visibility hypergraph, we make all of the arcs begin at the positive $x$-axis and move counterclockwise.

**Theorem 3.1.** *The maximum number of edges in a semi-arc $k$-visibility hypergraph with $n$ vertices is $n(k+2) - \frac{(k+1)(k+2)}{2}$ for $n \geq 3k+5$.*

*Proof.* We begin by providing a construction for a semi-arc $k$-visibility hypergraph with $n \geq 3k+5$ vertices and $n(k+1) - \frac{(k+1)(k+2)}{2}$ edges.

Define the arcs $A_i$ from $(i+1, 0)$ to $\left(i+1, \frac{i+k+3}{n}\pi\right)$ for $0 \leq i \leq n-2k-4$, arcs $B_i$ from $(n-2k-2+i, 0)$ to $\left(n-2k-2+i, \frac{n+i+1}{n}\pi\right)$ for $0 \leq i \leq k+1$, and arcs $C_i$ from $(n-k+i, 0)$ to $\left(n-k+i, \frac{2n-k+i}{n}\pi\right)$ for $0 \leq i \leq k$. Note that the number of $A_i$ is $n-2k-3 \geq k+2$. These arcs are shown in Figure 11.

First, there are $n-k-1$ edges along the angle 0 which do not pass through the origin. All other edges must pass through the origin.

If an edge does not pass through any of the $A_i$, then it passes through $B_0$, $B_1$, …, $B_j$ as well as some $k+1-j$ consecutive arcs of $C_0$, $C_1$, …, $C_k$ for some $0 \leq j \leq k$. For each $j$, the number of possible edges is $j+1$, so the number of edges is $1+2+\cdots+(k+1) = \frac{(k+1)(k+2)}{2}$.

Otherwise, the edge passes through one of the $A_i$. In this case, the edge either passes through at least one of the $B_i$ or it does not pass through any of the $B_i$. If the edge passes through at least one of the $B_i$, then it must pass through $A_0$. Suppose the edge passes through $A_0$, $A_1$, …, $A_j$ for some $0 \leq j \leq k$. Then, it must pass through some $k+1-j$

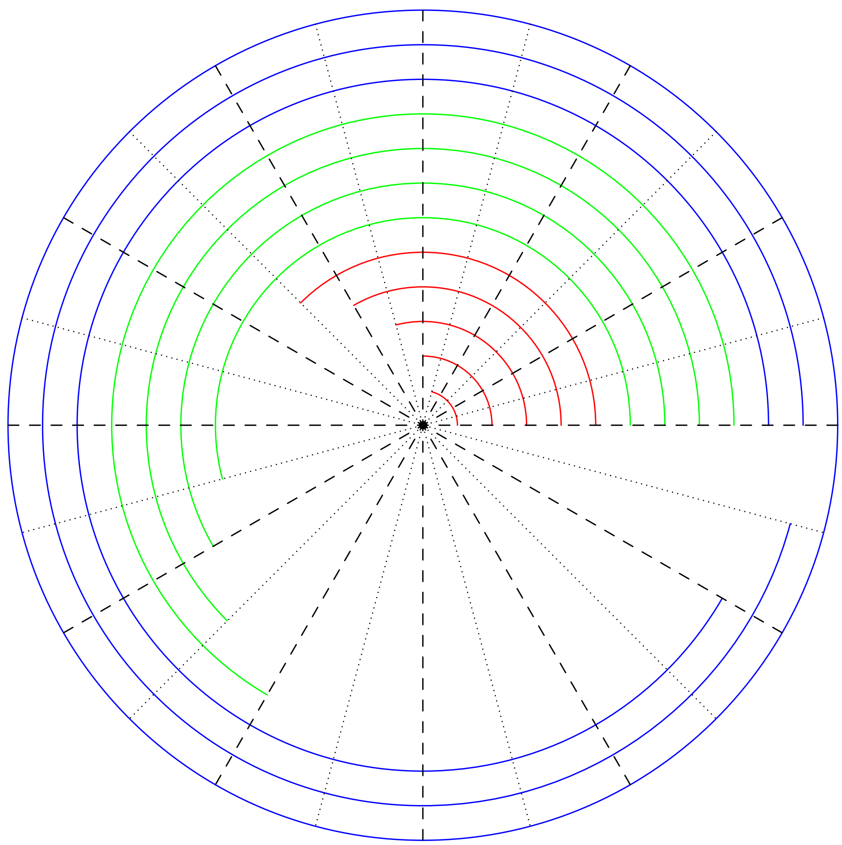

Figure 11: Optimal semi-arc $k$-visibility hypergraph on $n$ vertices for $n = 12$ and $k = 2$

| $k$ \ $n-k$ | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 1 | 3 | 6 | 9 | 11 | 13 | 15 | 17 | 19 | 21 | 23 | 25 | 27 |
| 1 | 1 | 4 | 8 | 13 | 17 | 21 | 24 | 27 | 30 | 33 | 36 | 39 | 42 |
| 2 | 1 | 5 | 10 | 16 | 23 | 28 | 33 | 38 | 42 | 46 | 50 | 54 | 58 |
| 3 | 1 | 6 | 12 | 19 | 27 | 35 | 42 | 48 | 54 | 60 | 65 | 70 | 75 |
| 4 | 1 | 7 | 14 | 22 | 31 | 41 | 50 | 58 | 66 | 73 | 80 | 87 | 94 |
| 5 | 1 | 8 | 16 | 25 | 35 | 46 | 57 | 68 | 77 | 86 | 95 | 103 | 111 |

Table 1: Table of lower bounds on the maximum number of edges in an semi-arc $k$-visibility hypergraph with $n$ vertices

consecutive arcs of $B_0$, $B_1$, ..., $B_{k+1}$, $C_0$, $C_1$, ..., $C_k$, starting from one of the $B_i$. For each $j$, there are $k+2$ edges, so there are a total of $(k+1)(k+2)$ edges in this case. Otherwise, the edge does not pass through any of the $B_i$, so it must intersect $C_0$, $C_1$, ..., $C_j$ and some $k+1-j$ consecutive arcs of $A_0$, $A_1$, ..., $A_{n-2k-4}$, $B_0$, $B_1$, ..., $B_{k+1}$ starting at one of the $A_i$ for some $0 \le j \le k$. For each $j$, there are $n-2k-3$ edges, so the total number of edges in this case is $(k+1)(n-2k-3)$.

Therefore, as all of these edges are distinct, the total number of edges is

$$\begin{aligned} & n-k-1+\frac{(k+1)(k+2)}{2}+(k+1)(k+2)+(k+1)(n-2k-3) \\ =& n+(k+1)n+(k+1)\left(-1+\frac{k+2}{2}+k+2-2k-3\right) \\ =& (k+2)n-\frac{(k+1)(k+2)}{2}. \end{aligned}$$

Now, we show that every semi-arc $k$-visibility hypergraph with $n$ vertices has at most $(k+2)n-\frac{(k+1)(k+2)}{2}$ edges.

In the arc visibility representation of the graph, draw each edge as a line segment that intersects exactly $k+2$ distinct arcs and is collinear with the origin. Rotate each edge counterclockwise until it hits an endpoint of an arc. For each arc, the number of edges at its counterclockwise endpoint is at most $k+2$. In addition, if an arc has the $i$th largest radius, then the number of edges is at most $i$. Therefore, the number of edges is at most

$$1+2+\cdots+(k+2)+(k+2)(n-k-2)=n(k+2)-\frac{(k+1)(k+2)}{2}.$$

□

For $n \le 3k+4$, we conjecture that the upper bound is not optimal. Table 1 shows our know lower bounds for the maximum number of edges of a semi-arc $k$-visibility hypergraph on $n$ vertices.

Now, we prove an optimal bound for the number of edges in arc $k$-visibility hypergraphs.

**Theorem 3.2.** *The number of edges in an arc $k$-visibility hypergraph with $n$ vertices is at most $n(2k+3)-(k+1)(k+3)$.*

*Proof.* Let the arcs be $A_1$, $A_2$, …, $A_n$ in decreasing order of radius, where $A_1$ has the largest radius and $A_n$ has the smallest radius. Rotate each edge counterclockwise until it hits an endpoint of an arc. For each arc, the number of edges at its counterclockwise endpoint is at most $k+2$, and the number of edges at its clockwise endpoint is at most $k+1$. In addition, if an arc has the $i$th largest radius, then the number of edges at its clockwise endpoint is at most $i$ and the number of edges at its counterclockwise endpoint is at most $i-1$. Define $a_i$ to be the number of edges at the counterclockwise endpoint of $A_i$ and $b_i$ to be the number of edges at the clockwise endpoint of $A_i$. Then, $a_i \leq \min(i, k+2)$ and $b_i \leq \min(i-1, k+1)$. The total number of edges is at most

$$\begin{aligned}&a_1+a_2+\cdots+a_n+b_1+b_2+\cdots+b_n\\ \leq&(1+0)+(2+1)+\cdots+(k+2+k+1)+(n-k-2)(2k+3)\\ =&n(2k+3)-(k+1)(k+2).\end{aligned}$$

Let $1 \leq i \leq k+1$. Consider the two edges that pass through either the clockwise or the counterclockwise endpoint of arc $A_i$ and intersect the outermost possible arc. If $a_i \neq i$ or $b_i \neq i-1$, then we decrease the upper bound for either $a_i$ or $b_i$ by 1. Otherwise, then the edge through the counterclockwise endpoint passes through arcs $A_1$, $A_2$, …, $A_i$, $B_1$, $B_2$, …, $B_{k+2-i}$, where the radii of arcs $B_1$, $B_2$, …, $B_{k+2-i}$ are in decreasing order. If we rotate this segment slightly counterclockwise, then it intersects $A_1$, $A_2$, …, $A_{i-1}$, $B_1$, $B_2$, …, $B_{k+2-i}$, but not $A_i$. Now, rotate this segment counterclockwise until the counterclockwise endpoint of one of the $B_j$ is reached, or the segment intersects the clockwise endpoint of another arc that is not $A_1$, $A_2$, …, $A_{i-1}$. If this arc is $A_i$, then the edge consisting of $A_1$, $A_2$, …, $A_i$, $B_1$, $B_2$, …, $B_{k+2-i}$ is counted twice, so we can subtract 1 from the total upper bound. If this arc is $B_j$ for some $j$ and $x-1$ of the arcs $A_1$, $A_2$, …, $A_{i-1}$ currently do not intersect the segment, then there are at most $i+j-x-1$ arcs directly above the counterclockwise endpoint. If $B_j = A_m$, then $m \geq i+j$, so we decrease the upper bound for $a_m$ by 1. Otherwise, if this arc is not $A_i$ or any of the $B_j$, then suppose it is $A_m$. It must hit the clockwise endpoint of $A_m$, and $A_m$ must be between $B_j$ and $B_{j+1}$ for some $0 \leq j \leq k+1-i$, where $B_0 = A_i$. Let $y-1$ be the number of arcs $A_1$, $A_2$, …, $A_{i-1}$ which are not directly above the clockwise endpoint of $A_m$. As $m \geq i+j+1$ and there are at most $i+j-y-1$ arcs above the clockwise endpoint of $A_m$, we decrease the upper bound of $b_m$ by 1.

We need to verify that we do not overcount decreases in the upper bound. First, we show that the edges that are counted multiple times in the sum have been subtracted the correct number of times. Suppose some edge $B_1$, $B_2$, …, $B_{k+2}$ is subtracted multiple times. This edge is subtracted each time that there exists some $B_j$ where $A_i = B_i$ for $i \leq j \leq k+1$, the edge can be drawn at both the clockwise and counterclockwise endpoints of $B_j$, and the arc $B_i$ exists in the entire space between these two endpoints for $i > j$. Let these values of $j$ be $j_1$, $j_2$, …, $j_m$. Then, the angles where the arcs $B_{j_1}$, $B_{j_2}$, …, $B_{j_m}$, and $B_{k+2}$ do not exist are all disjoint, while the edge can be drawn on both the clockwise and counterclockwise endpoints of each of the arcs $B_j$, so rotating the edges drawn on the clockwise endpoints in the counterclockwise direction will eventually create another location where the edge appears. Therefore, the edge is counted at least $m + 1$ times, while the edge is subtracted $m$ times.

Now, we show that our decreases in the upper bounds of $a_m$ and $b_m$ are not overcounted. Suppose $A_m$ is an arc where either $a_m$ or $b_m$ has been decreased in the previous process. Let $x$ be the number of arcs $A_i$ for $i \leq \min(k+1, m-1)$ that are not directly above the counterclockwise endpoint of $A_m$. If $a_m$ is decreased for a given value of $i$, then $A_i$ must be one of these $x$ arcs. Take the largest such $i$. Then, $a_m \leq m - x - 1$, which is the value of the upper bound after it has been decreased for each of the $x$ values of $i$ and possibly at $i = m$. Similarly, let $y$ be the number of arcs $A_i$ for $i \leq \min(k+1, m-1)$ that are not directly above the clockwise endpoint of $A_m$. If $b_m$ is decreased for a given value of $i$, then $A_i$ must be one of these $y$ arcs. Take the largest such $i$. Then, $b_m \leq i + j - y - 2$, which is the value of the upper bound after it has been decreased for each of the $y$ values of $i$ and possibly at $i = m$.

Therefore, since we lowered the upper bound by 1 a total of $k + 1$ times, we obtain the upper bound of $n(2k + 3) - (k + 1)(k + 3)$. □

Now, we give a construction which achieves this bound for $n \geq 2k + 5$.

**Lemma 3.3.** *Suppose an arc $k$-visibility hypergraph on $n$ vertices contains $x$ edges. If some ray from the origin contains an endpoint of an arc and intersects $k+1$ other arcs in both directions from that arc, then there exists an arc $k$-visibility hypergraph on $n+1$ vertices containing $x + 2k + 3$ edges.*

*Proof.* Draw the arcs and the ray. Consider the arc that has an endpoint on the ray. Draw a new arc that crosses the ray with very small length very close to that arc, picking the direction such that there are at least $k+1$ arcs on either direction of the new arc along the ray. Every edge in the original graph still exists in the new graph, and the new arc adds $(k + 1) + (k + 1) + 1 = 2k + 3$ edges containing that arc. □

**Theorem 3.4.** *For $k \geq 3$ and $n \geq 2k + 5$, there exists an arc $k$-visibility hypergraph on $n$ vertices with exactly $n(2k + 3) - (k + 1)(k + 3)$ edges.*

*Proof.* Construct the following arcs:

- $A_0$ from $\left(1, (2n - k - 4)\frac{\pi}{n}\right)$ to $\left(1, \frac{\pi}{n}\right)$,
- $A_1$ from $\left(2, (2n - k - 2)\frac{\pi}{n}\right)$ to $\left(2, (n - k - 4)\frac{\pi}{n}\right)$,

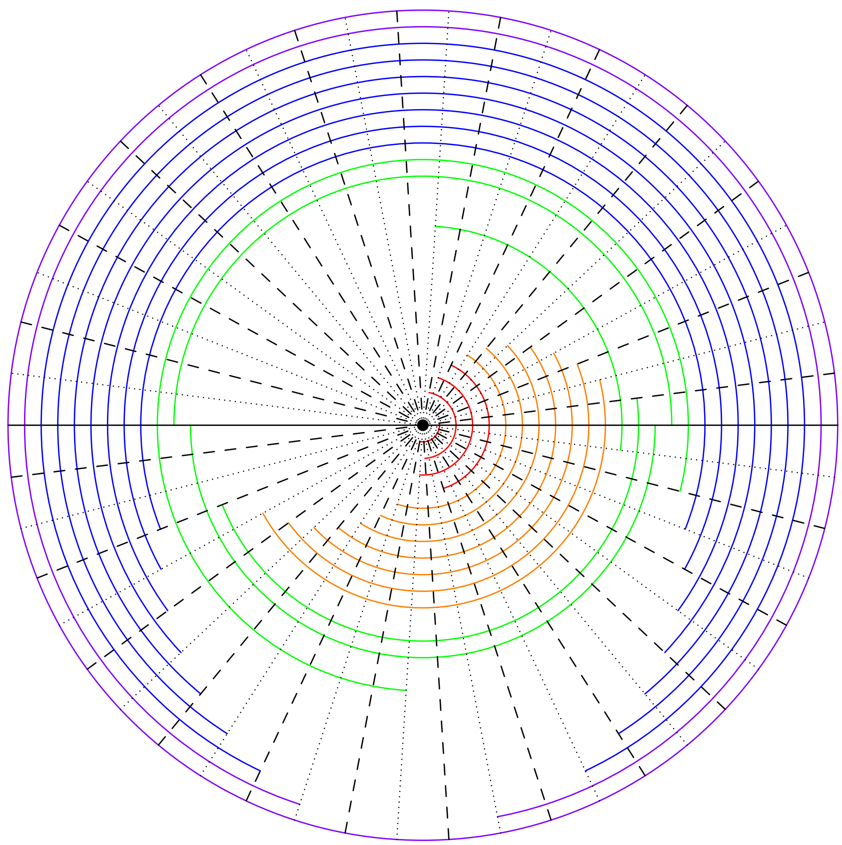

Figure 12: Optimal arc $k$-visibility hypergraph on $n$ vertices for $n = 25$, $k = 10$

- $A_2$ from $\big(3, (2n-k-3)\frac{\pi}{n}\big)$ to $\big(3, (n-k-5)\frac{\pi}{n}\big)$,
- $A_3$ from $\big(4, (2n-k)\frac{\pi}{n}\big)$ to $\big(4, (n-k-6)\frac{\pi}{n}\big)$,
- $A_i$ from $\big(i+1, (2n-k-1-i)\frac{\pi}{n}\big)$ to $\big(i+1, (n-k-3-i)\frac{\pi}{n}\big)$ for $4 \le i \le n-k-5$,
- $B_0$ from $\big(n-k-3, (2n-1)\frac{\pi}{n}\big)$ to $\big(n-k-3, (n-k-3)\frac{\pi}{n}\big)$
- $B_1$ from $\big(n-k-2, (n+3)\frac{\pi}{n}\big)$ to $\big(n-k-2, \frac{\pi}{2n}\big)$
- $B_2$ from $(n-k-1, \pi)$ to $(n-k-2, 0)$
- $B_3$ from $(n-k, 0)$ to $(n-k, \pi)$,
- $B_4$ from $\big(n-k+1, (2n-2)\frac{\pi}{n}\big)$ to $\big(n-k+1, (2n-k-3)\frac{\pi}{n}\big)$,
- $C_i$ from $\big(n-k+2+i, (2n-3-i)\frac{\pi}{n}\big)$ to $\big(n-k+2+i, (n+3+i)\frac{\pi}{n}\big)$ for $0 \le i \le k-4$,
- $C_{k-3}$ from $\big(n-1, (2n-k-1)\frac{\pi}{n}\big)$ to $\big(n-1, (2n-k-5)\frac{\pi}{n}\big)$, and
- $C_{k-2}$ from $(n, 0)$ to $(n, 2\pi)$.

There are $n-k-4 = k+1$ arcs $A_i$ and $k-1$ arcs $C_i$. We will list out $3(k+2)^2$ distinct edges in counterclockwise order. First, we count the edges that do not intersect the origin.

- There are $n-k-1$ distinct edges along the ray at angle 0.
- There are $k+1$ edges which contain $B_1$ and $B_3$ but not $B_2$ at angle $\frac{\pi}{n}$.
- There are $\frac{k(k+1)}{2}$ edges which contain $B_0$, $B_3$, some interval of edges $A_i$ to $A_j$ for $1 \le i \le j \le k$, then the first $k-1-j+i$ edges of $B_4$ and the $C_i$.
- There is 1 edge containing $B_0$, $B_3$, $B_4$, and all of the $C_i$.
- There are $k-3$ distinct edges containing the edges from $A_j$ to $A_k$, $B_1$, $B_2$, $B_4$, and the edges from $C_{k-j+1}$ to $C_{k-2}$ for $4 \le j \le k$.
- There is 1 edge containing $A_0$, $A_4$ through $A_k$, $B_1$, $B_2$, $B_4$, and $C_{k-2}$
- There is 1 edge containing $A_0$, $A_2$, $A_4$ through $A_k$, $B_1$, $B_2$, and $B_4$.
- There is 1 edge containing $A_2$, $A_4$ through $A_k$, $B_1$, $B_2$, $B_4$, and $C_{k-2}$.
- There is 1 edge containing $A_0$, $A_1$, $A_2$, $A_4$ through $A_k$, $B_1$, and $B_2$.
- There are 2 edges containing $A_1$, $A_2$, $A_4$ through $A_k$, $B_1$, $B_2$, and one of $C_{k-3}$ or $C_{k-2}$.
- There is 1 edge containing $A_2$, $A_4$ through $A_k$, $B_1$, $B_2$, $C_{k-3}$, and $C_{k-2}$.

- There are 2 edges containing $A_1$ through $A_k$, $B_1$, and one of $A_0$ and $B_2$.
- There are $\frac{k(k-1)}{2}$ edges containing $A_i$ through $A_k$, $B_1$, $B_2$, and $C_j$ through $C_{j+i-2}$ for $2 \leq i \leq k$ and $0 \leq j \leq k-i$.
- There are $k$ edges containing $A_i$ through $A_k$, $B_1$, $B_2$, $B_4$, and $C_0$ through $C_{i-3}$ for $2 \leq i \leq k+1$.
- There are $k-1$ edges at angle $2\pi - \frac{\pi}{n}$ which contain $B_0$ and $B_4$.

The total number of edges which do not pass through the origin is $k^2 + 5k + 11$. Now, we count all edges that pass through the origin.

- At angles 0 and $\pi$, there are $k+1$ edges which intersect the origin.
- At angles $\frac{\pi}{n}$ and $\pi + \frac{\pi}{n}$, there are $k$ edges which intersect $B_2$, $B_4$, and the origin.
- At angles $\frac{2\pi}{n}$ and $\pi + \frac{2\pi}{n}$, there are $k+1$ edges.
- At angles $\frac{2.5\pi}{n}$ and $\pi + \frac{2.5\pi}{n}$, there are 2 edges that contain $A_{k-1}$ and $B_0$ but not $A_k$.
- For $0 \leq i \leq j \leq k-2$, there exist $(k-2)(k-1)$ edges intersecting $C_j$ through $C_i$, $B_4$, $B_2$, $B_1$, $A_k$ through $A_{k+1-i}$ or $A_{k+2-i}$, and $A_2$ through $A_{k-j-1}$ or $A_{k-j}$ (excluding the second case when $i = 0$ and the first case when $j = k-2$).
- There are $(k-2)(k-3)$ edges intersecting $A_j$ through $A_i$, $A_1$ through $A_{i-2}$ or $A_{i-3}$, $B_0$, $B_3$, $B_4$, and $C_0$ through $C_{k-j-1}$ or $C_{k-j}$ for some $4 \leq i \leq j \leq k$.
- For $4 \leq i \leq k+1$, there are $3(k-2)$ edges intersecting $A_k$ through $A_i$, $A_1$ through $A_{i-2}$, then either the triples $B_4$, $B_2$, and $B_1$, $B_2$, $B_1$, and $B_0$, or $B_1$, $B_0$, and $B_3$.
- Similarly, for $4 \leq i \leq k+1$, there are $3(k-2)$ edges intersecting $A_k$ through $A_i$, $A_1$ through $A_{i-3}$, and some quadruple of four consecutive arcs of $B_4$, $B_2$, $B_1$, $B_0$, $B_3$, and $B_4$.
- There are $k+1$ edges which intersect some $k+2$ consecutive arcs of $C_{k-2}$, $B_4$, $B_2$, $B_1$, $A_k$ through $A_4$, $A_0$, $A_1$, $B_0$, $B_3$, $B_4$, and $C_0$ through $C_{k-2}$ including both $A_0$ and $A_1$
- There are $k+1$ more edges after a small rotation that removes $A_1$ such that the edge now includes $A_0$ and $B_0$.
- After one more rotation, $A_2$ is added, so there are $k$ more edges including $A_2$, $A_0$, and $B_0$
- After another rotation, $B_4$ and $B_0$ are removed, adding $k+1$ edges that include both $A_0$ and $B_3$.

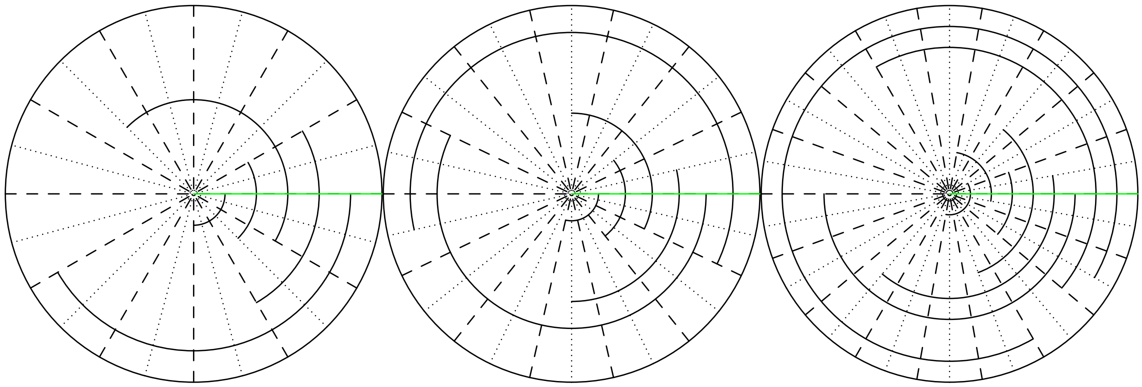

Figure 13: Optimal constructions for $k = 0$, $n = 6$, 15 edges, $k = 1$, $n = 7$, 27 edges, and $k = 2$, $n = 9$, 48 edges

- Now, we have $k - 3$ more edges intersecting $A_i$ through $A_4$, $A_2$, $A_1$, $A_0$, $B_3$, $B_4$, and $C_0$ through $C_{k-i-1}$ for $4 \leq i \leq k$
- There is 1 edge intersecting $B_1$, $A_k$ through $A_4$, $A_2$, $A_1$, $A_0$, and $B_3$
- There are $k$ edges intersecting $A_i$ through $A_0$, $B_3$, $B_4$, and $C_0$ through $C_{k-i-2}$ for $1 \leq i \leq k$, as the case of $A_0$, $B_3$, $B_4$, and $C_0$ through $C_{k-2}$ was already counted earlier.

The number of edges that pass through the origin is $2k^2 + 7k + 1$, so the total number of edges is $3k^2 + 12k + 12 = n(2k + 3) - (k + 1)(k + 3)$. □

We have proven the bound is optimal for $k \geq 3$ and $n \geq 2k+5$. In Figure 13, we provide constructions that prove the bound is optimal for $k = 0, 1, 2$ and $n \geq \max(6, 2k + 5)$.

We do not know the optimal bound when $k + 4 \leq n \leq 2k + 4$. We conjecture that Theorem 3.2 is never optimal for $n \leq 2k + 3$. We show our known lower bounds in Table 2. Note that for $k = 3, 4, 5$ and $n = 2k + 4$, our lower bound is equal to the upper bound of $n(2k + 3) - (k + 1)(k + 3)$ given in Theorem 3.2. We conjecture that there exists such a construction for all $k \geq 3$ and $n = 2k + 4$ having exactly $n(2k + 3) - (k + 1)(k + 3)$ edges.

## 4 Arc $k$-visibility graph

Sawhney and Weed [14] proved that the maximum number of edges in an arc $k$-visibility graph is upper bounded by $3n(k + 1) - \frac{3(k+1)(k+2)}{2}$. This upper bound was proven optimal for $k = 0$ and $n \geq 6$ in Corollary 4 in [14]. In this section, we show that this upper bound is sharp for $k \geq 1$ and $n \geq 7k + 6$. We split the construction into three parts: $k = 1$, $k = 2$, and $k \geq 3$.

We start with the construction for $k = 1$.

**Theorem 4.1.** *For $n \geq 13$, there exists an arc 1-visibility graph with $6n - 9$ edges.*

| $k$ \ $n-k$ | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 1 | 3 | 6 | 10 | 15 | 18 | 21 | 24 | 27 | 30 | 33 | 36 |
| 1 | 1 | 4 | 10 | 20 | 27 | 32 | 37 | 42 | 47 | 52 | 57 | 62 |
| 2 | 1 | 5 | 15 | 29 | 41 | 48 | 55 | 62 | 69 | 76 | 83 | 90 |
| 3 | 1 | 6 | 20 | 38 | 55 | 66 | 75 | 84 | 93 | 102 | 111 | 120 |
| 4 | 1 | 7 | 25 | 47 | 67 | 85 | 97 | 108 | 119 | 130 | 141 | 152 |
| 5 | 1 | 8 | 30 | 55 | 79 | 102 | 120 | 134 | 147 | 160 | 173 | 186 |
| 6 | 1 | 9 | 34 | 63 | 92 | 119 | 141 | 161 | 177 | 192 | 207 | 222 |
| 7 | 1 | 10 | 38 | 70 | 102 | 136 | 162 | 187 | 208 | 226 | 243 | 260 |
| 8 | 1 | 11 | 42 | 78 | 114 | 150 | 184 | 213 | 238 | 260 | 281 | 300 |
| 9 | 1 | 12 | 46 | 85 | 126 | 164 | 203 | 237 | 267 | 295 | 320 | 342 |

Table 2: Table of lower bounds on the maximum number of edges in an arc $k$-visibility hypergraph with $n$ vertices

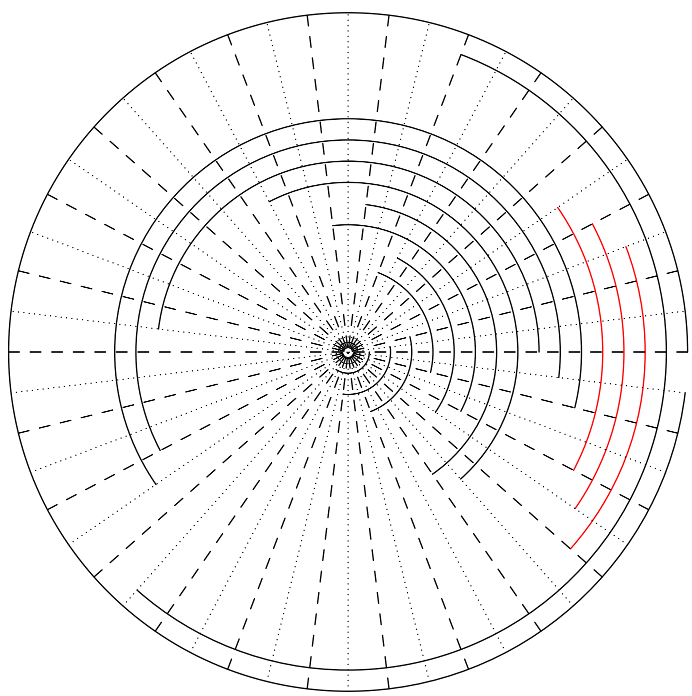

Figure 14: Optimal arc $k$-visibility graph on $n$ vertices for $n = 16$ and $k = 1$

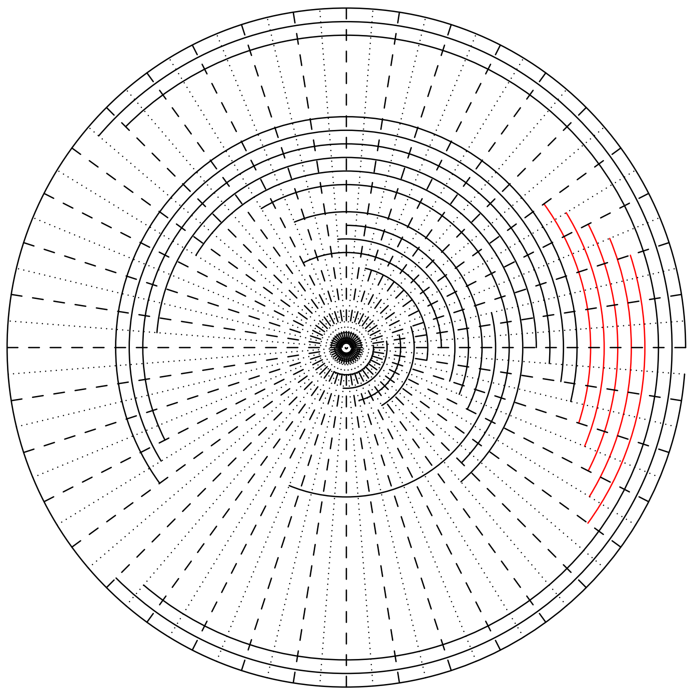

Figure 15: Optimal arc $k$-visibility graph on $n$ vertices for $n = 25$ and $k = 2$

*Proof.* Fix the angles of the arcs at radii 1, 2, ..., 11, $n-1$, and $n$ as shown in Figure 14. Then, add $n-13$ arcs with counterclockwise endpoint between angles 0 and $\frac{\pi}{4}$ and clockwise endpoint between angles $\frac{23}{13}\pi$ and $\frac{25}{13}\pi$, where the arcs with larger radii have endpoints at smaller angles. There are 69 edges between the arcs with radii 1, 2, ..., 11, $n-1$, and $n$, and each additional arc added increases the number of edges by 6, with the arc of radius $i$ having visibilities to the arcs with radii 7, 8, $i-2$, $i-1$, $n-1$, and $n$. Therefore, there are $6n-9$ edges. □

Now, we provide the construction for $k = 2$.

**Theorem 4.2.** *For $n \geq 20$, there exists an arc 2-visibility graph with $9n-18$ edges.*

*Proof.* Fix the angles of the arcs at radii 1, 2, ..., 17, $n-2$, $n-1$, and $n$ as shown in 15. Add $n-20$ arcs with counterclockwise endpoint between angles 0 and $\frac{\pi}{4}$ and clockwise endpoint between angles $\frac{9}{5}\pi$ and $\frac{19}{10}\pi$, where the arcs with larger radii have endpoints at smaller angles. There are 162 edges between the arcs with radii 1, 2, ..., 17, $n-2$, $n-1$, and $n$, and each additional arc added increases the number of edges by 9, with the arc of radius $i$ having visibilities to the arcs with radii 11, 12, 13, $i-3$, $i-2$, $i-1$, $n-2$, $n-1$, and $n$. Therefore, there are $9n-18$ edges. □

Now, we provide the construction for $k \geq 3$.

**Theorem 4.3.** *For $k \geq 3$ and $n \geq 7k+6$, there exists an arc $k$-visibility graph with $3n(k+1) - \frac{3(k+1)(k+2)}{2}$ edges.*

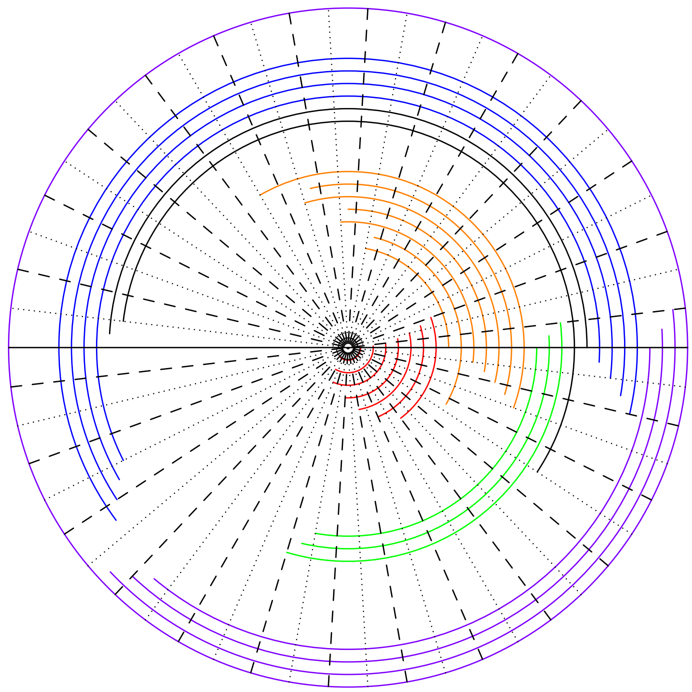

Figure 16: Optimal arc $k$-visibility graph on $n$ vertices for $n = 27$ and $k = 3$

*Proof.* Let $n \geq 7k + 6$ and $k \geq 3$. Construct the following arcs:

- $A_i$ from $\left(i+1, (2n+4k+4+2i)\frac{\pi}{2n}\right)$ to $\left(i+1, i\frac{\pi}{2n}\right)$ for $0 \leq i \leq k-1$,
- $A_i$ from $\left(i+1, (2n+2k+4+4i+(n-6k-5))\frac{\pi}{2n}\right)$ for $k \leq i \leq 2k$,
- $B_i$ from $\left(i+2k+2, (4n-i)\frac{\pi}{2n}\right)$ to $\left(i+2k+2, (2i+n-3)\frac{\pi}{2n}\right)$ for even $0 \leq i \leq 2k$,
- $B_1$ from $\left(2k+3, (4n-2k-3)\frac{\pi}{2n}\right)$ to $\left(2k+3, (n-4)\frac{\pi}{2n}\right)$,
- $B_i$ from $\left(i+2k+2, (4n-i)\frac{\pi}{2n}\right)$ to $\left(i+2k+2, (2i+n-6)\frac{\pi}{2n}\right)$ for odd $3 \leq i \leq 2k$,
- $C_i$ from $\left(i+4k+3, (3n-3-i)\frac{\pi}{2n}\right)$ to $\left(i+4k+3, i\frac{\pi}{2n}\right)$ for $0 \leq i \leq n-6k-7$
- $D_0$ from $\left(n-2k-3, (4n-2k-4)\frac{\pi}{2n}\right)$ to $\left(n-2k-3, (2n-2)\frac{\pi}{2n}\right)$,
- $D_1$ from $(n-2k-2, 0)$ to $\left(n-2k-2, (2n-1)\frac{\pi}{2n}\right)$,
- $E_i$ from $\left(i+n-2k-1, (4n-1-i)\frac{\pi}{2n}\right)$ to $\left(i+n-2k-1, (2n+2k+2+i)\frac{\pi}{2n}\right)$ for $0 \leq i \leq k$,
- $F_i$ from $\left(i+n-k, (2n+4k+3-i)\frac{\pi}{2n}\right)$ to $\left(i+n-k, i\frac{\pi}{2n}\right)$ for $0 \leq i \leq k-1$, and
- $F_k$ from $(n, 0)$ to $(n, 2\pi)$.

The construction for $n = 27$ and $k = 3$ is shown in Figure 16. Now, we will show that this arc $k$-visibility graph has $3n(k+1) - \frac{3(k+1)(k+2)}{2}$ edges. We list all of the edges. First, we start with the edges containing some arc $A_i$.

- $A_i$ and $A_j$ for $0 \leq i < j \leq 2k$, where $j-i \leq k+1$. There are $k(k+1)+\frac{k(k+1)}{2} = \frac{3k(k+1)}{2}$ such edges.

- $A_i$ and $B_j$ for $0 \leq i \leq k-1$ and $0 \leq j \leq 2k$ and $i \leq k-1$. There are $k(2k+1)$ such edges.

- $A_k$ and $B_{2j}$ for $1 \leq j \leq k$. There are $k$ such edges.

- $A_k$ and $B_j$ for $0 \leq j \leq 1$. There are 2 such edges.

- $A_i$ and $B_j$ for $k+1 \leq i \leq 2k$ and $0 \leq j \leq 2k$. There are $k(2k+1)$ such edges.

- $A_i$ and $C_j$ for $0 \leq i \leq k-1$ and $0 \leq j \leq n-6k-7$. There are $k(n-6k-6)$ such edges.

- $A_i$ and $C_j$ for $k \leq i \leq 2k$ and $0 \leq j \leq k$, where we set $C_k = F_0$ if $n = 7k+6$. There are $(k+1)^2$ such edges.

- $A_i$ and $D_j$ for $0 \leq i \leq k$ and $0 \leq j \leq 1$. There are $2(k+1)$ such edges.

- $A_i$ and $E_j$ for $0 \leq i \leq 2k$ and $0 \leq j \leq k$. There are $(2k+1)(k+1)$ such edges.

- $A_i$ and $F_j$ for $0 \leq i \leq k-1$ and $0 \leq j \leq k$. There are $k(k+1)$ such edges.

There are $kn + \frac{7k^2+13k+12}{2}$ edges in these cases.
The following edges contain some $B_i$ but do not contain any arc $A_i$.

- $B_{2i}$ and $B_j$ for $0 \leq i \leq k-1$ and $2i+1 \leq j \leq 2i+k+2$, where we set $B_{2k+m} = D_{m-1}$ for $1 \leq m \leq 2$ and $B_{2k+m} = E_{m-3}$ for $m \geq 3$. There are $k(k+2)$ such edges.

- $B_{2i+1}$ and $B_j$ for $0 \leq i \leq k-1$ and $2i+2 \leq j \leq 2i+k+2$, where we set $B_{2k+m} = D_{m-1}$ for $1 \leq m \leq 2$ and $B_{2k+m} = E_{m-3}$ for $m \geq 3$. There are $k(k+1)$ such edges.

- $B_1$ and $B_j$ for $k+3 \leq i \leq 2k$. There are $k-2$ such edges.

- $B_{2k}$ and $D_j$ for $0 \leq j \leq 1$. There are 2 such edges.

- $B_{2k}$ and $E_j$ for $0 \leq j \leq k-2$. There are $k-1$ such edges.

- $B_0$ and $C_j$ for $i = 0, 2$ and $0 \leq j \leq n-6k-7$. There are $n-6k-6$ such edges.

- $B_2$ and $C_0$, which is 1 edge.

- $B_1$ and $C_j$ for $0 \leq j \leq k$, where we set $C_k = D_0$ if $n = 7k+6$. There are $k+1$ such edges.

- $B_i$ and $C_j$ for $k \leq i \leq 2k$ and $0 \leq j \leq n-6k-7$. There are $(k+1)(n-6k-6)$ such edges.

- $B_i$ and $E_j$ for $0 \le i \le k$ and $0 \le j \le k$. There are $(k+1)^2$ such edges.
- $B_i$ and $F_j$ for $0 \le i \le k$ and $0 \le j \le k$. There are $(k+1)^2$ such edges.

There are $(k+2)n - 2k^2 - 8k - 9$ edges in these cases.

The following arcs contain some $C_i$ but do not contain any $A_i$ or $B_i$.

- $C_i$ and $C_j$ for $0 \le i < j \le n-6k-7$, where $j - i \le k+1$. There are $(n-7k-7)(k+1) + \frac{k(k+1)}{2} = \frac{(2n-13k-14)(k+1)}{2}$ such edges.
- $C_i$ and $D_0$ for $n-7k-7 \le i \le n-6k-7$, where we set $C_{-1} = A_{2k}$ if $n = 7k+6$. There are $k+1$ such edges.
- $C_i$ and $D_1$ for $n-7k-6 \le i \le n-6k-7$. There are $k$ such edges.
- $C_i$ and $E_j$ for $n-7k-6 \le i \le n-6k-7$ and $0 \le j \le k$. There are $k(k+1)$ such edges.
- $C_i$ and $F_j$ for $(n-6k-7-i)+j \le k$, where we set $C_{-1} = A_{k-1}$ if $n = 7k+6$. There are $\frac{(k+1)(k+2)}{2}$ such edges.

There are $(k+1)n - 5k^2 - 9k - 5$ edges in these cases.

Finally, the following edges contain only arcs $D_i$, $E_i$, and $F_i$.

- $D_0$ and $D_1$, which is 1 edge.
- $D_0$ and $E_j$ for $0 \le j \le k$. There are $k+1$ such edges.
- $D_1$ and $E_j$ for $0 \le j \le k$. There are $k+1$ such edges.
- $D_0$ and $F_j$ for $0 \le j \le k$. There are $k+1$ such edges.
- $E_i$ and $E_j$ for $0 \le i < j \le k$. There are $\frac{k(k+1)}{2}$ such edges.
- $E_i$ and $F_j$ for $0 \le i \le k$ and $0 \le j \le k$. There are $(k+1)^2$ such edges.
- $F_i$ and $F_j$ for $0 \le i < j \le k$. There are $\frac{k(k+1)}{2}$ such edges.

There are $2k^2 + 6k + 5$ edges in these cases.

The sum of the number of edges in all cases is

$$kn + \frac{7k^2 + 13k + 12}{2} + (k+2)n - 2k^2 - 8k - 9 + (k+1)n - 5k^2 - 9k - 5 + 2k^2 + 6k + 5,$$

which is equal to $(3k+3)n - \frac{3(k+1)(k+2)}{2}$. □

## 5 Rectangle $k$-visibility hypergraphs

We prove upper and lower bounds on the maximum number of edges in a rectangle $k$-visibility hypergraph with $n$ vertices. Similar to the section on bar $k$-visibility hypergraphs, we assume that no two rectangles have collinear edges. As the vertical edges of any rectangle $k$-visibility hypergraph form a bar $k$-visibility hypergraph, the number of vertical edges is upper bounded by $(2k+3)n - 3k^2 - 8k - 6$ by Theorem 2. Similarly, the number of horizontal edges is at most $(2k+3)n - 3k^2 - 8k - 6$, so the total number of edges is at most $(4k+6)n - 6k^2 - 16k - 12$. However, this upper bound can be improved. The optimal bound for $k = 0$ was shown to be $6n - 20$ in [11]. No other bounds are known for any nonzero value of $k$. We provide upper and lower bounds for the maximum number of edges in a rectangle $k$-visibility hypergraph on $n$ vertices for all $k$ and $n \geq 4k + 8$. However, our bound in Theorem 5.1 is not optimal even for $k = 0$ as it only obtains an upper bound of $6n - 11$ for $k = 0$.

**Theorem 5.1.** *The maximum number of edges in a rectangle $k$-visibility hypergraph with $n$ vertices is at most $(4k+6)n - (k+1)(9k+11)$ for $n \geq 4k + 4$.*

*Proof.* Represent the edges in the rectangle $k$-visibility graph as horizontal or vertical line segments which have both endpoints on two rectangles and intersect exactly $k$ other rectangles. Shift each vertical edge to the left and shift each horizontal edge up until the edges hit a side of some rectangle.

Label the vertical rectangle sides from 1 to $2n$ from left to right. Consider the $i$th vertical side. Define $a_i$ to be the minimum of $k+1$ and the number of complete rectangles to the left of this side, define $b_i$ to be the minimum of $k+1$ and the number of complete rectangles to the right of this side, and define $c_i$ to be the minimum of $2k+2$ and the number of other rectangles that intersect the line containing this side.

If this side is the left side of a rectangle, then the total number of edges at the left and top sides of that rectangle is at most $\max(0, c_i - k) + \min(a_i, b_i) + 1$. Otherwise, if this side is the right side of a rectangle, then the total number of edges at the right and bottom sides of that rectangle is at most $\max(0, c_i - k - 1) + \min(a_i, b_i)$. We must have $2a_i + c_i \leq i - 1$ and $2b_i + c_i \leq 2n - i$.

Notice that $|c_{i+1} - c_i| \leq 1$ for all $1 \leq i \leq 2n - 1$ as the set of rectangles intersecting a vertical line through the $i$th side is the same as the set of rectangles intersecting a vertical line through the $i+1$th side, except for possibly deleting the rectangle through the $i$th side and adding the rectangle through the $i+1$th side.

To prove the upper bound, we maximize the sum of our upper bound expressions for $1 \leq i \leq 2n$ over all sequences of nonnegative integers $0 \leq a_i \leq k+1$, $0 \leq b_i \leq k+1$, and $0 \leq c_i \leq 2k+2$ satisfying $2a_i + c_i \leq i - 1$, $2b_i + c_i \leq 2n - i$, and $|c_{i+1} - c_i| \leq 1$, and we only use the fact that there are $n$ left sides and $n$ right sides.

For $2k+3 \leq i \leq 4k+4$, if $c_i < i - 2k - 3$, then we set $c_i = i - 2k - 3$. This change still satisfies $2a_i + c_i \leq i - 1$ as $a_i \leq k + 1$, and it satisfies $2b_i + c_i \leq 2n - i$ as $2n - i \geq 4k + 4$. We also need to show it satisfies $|c_{i+1} - c_i| \leq 1$.

If $c_i \geq i - 2k - 3$ originally, then $c_i$ remains unchanged. If $c_{i+1} < i - 2k - 2$ originally, then $i - 2k - 3 \leq c_i \leq i - 2k - 2$, so $|c_{i+1} - c_i| \leq 1$ is still true. Otherwise, $c_{i+1}$ is also unchanged, so $|c_{i+1} - c_i| \leq 1$ is still true.

Otherwise, if $c_i < i - 2k - 3$ originally, then $c_{i+1} < i - 2k - 2$ originally, so now $c_i$ becomes $i - 2k - 3$ and $c_{i+1}$ becomes $i - 2k - 2$, so $|c_{i+1} - c_i| \leq 1$ still holds.

As increasing $c_i$ increases our upper bound, we can assume $c_{4k+4} \geq 2k + 1$. Similarly, for $2n - 4k - 4 \leq i \leq 2n - 2k - 3$, if $c_i < 2n - i - 2k - 3$, then we set $c_i = 2n - i - 2k - 3$. This means we can get $c_{2n-4k-4} \geq 2k + 1$.

This implies that for $1 \leq i \leq 4k + 4$, the number of edges is at most

$$\begin{aligned}\max(0, c_i - k) + a_i + 1 &\leq \frac{i+1}{2} - k + \max\left(k - \frac{c_i}{2}, \frac{c_i}{2}\right) \\ &= (2k+4) - \left(3k + 4 - \frac{i+1}{2} - \max\left(k - \frac{c_i}{2}, \frac{c_i}{2}\right)\right)\end{aligned}$$

if the side is the left side of the rectangle, and at most

$$\begin{aligned}\max(0, c_i - k - 1) + a_i &\leq \max\left(k + 1 - \frac{c_i}{2}, \frac{c_i}{2}\right) + \frac{i-1}{2} - k - 1 \\ &= (2k+2) - \left(3k + 3 - \frac{i-1}{2} - \max\left(k + 1 - \frac{c_i}{2}, \frac{c_i}{2}\right)\right)\end{aligned}$$

if the side is the right side of the rectangle.

The second term of each expression is at least $2k+2-\frac{i-1}{2}+\min\left(\frac{c_i}{2}, k+1-\frac{c_i}{2}\right)$. Therefore, since $c_i$ can take on any value between 0 and $2k+1$, the sum of the second term of this expression for $1 \leq i \leq 4k+4$ is at least $(4k+5)(k+1)+\frac{(k+1)^2}{2}$. Similarly, the corresponding terms for $2n - 4k - 3 \leq i \leq 2n$ add up to the same number, and the number of edges for other $i$ are bounded above by $2k + 4$ for left and top sides and $2k + 2$ for right and bottom sides. Therefore, the total number of edges is upper bounded by $(4k+6)n-(k+1)(9k+11)$. □

Now, we provide a construction of a rectangle $k$-visibility hypergraph on $n$ vertices with a large number of edges based on the construction given in [11]. This construction is only optimal for $k = 0$.

**Theorem 5.2.** *There exists a rectangle $k$-visibility hypergraph on $n \geq 4k + 8$ vertices with $(4k + 6)n - 4(k + 1)(3k + 5)$ edges.*

*Proof.* First, we create an arrangement of $n - 4k - 4 \geq 4$ squares of side length 3. If $n-4k-4 = 2a$ is even, construct squares centered at $(4i, i)$ and $(4i-2, i+4)$ for $1 \leq i \leq a$. Otherwise, if $n - 4k - 4 = 2a + 1$ is odd, construct squares centered at $(4i, i)$ for $1 \leq i \leq a$ and $(4i - 2, i + 4)$ for $1 \leq i \leq a + 1$.

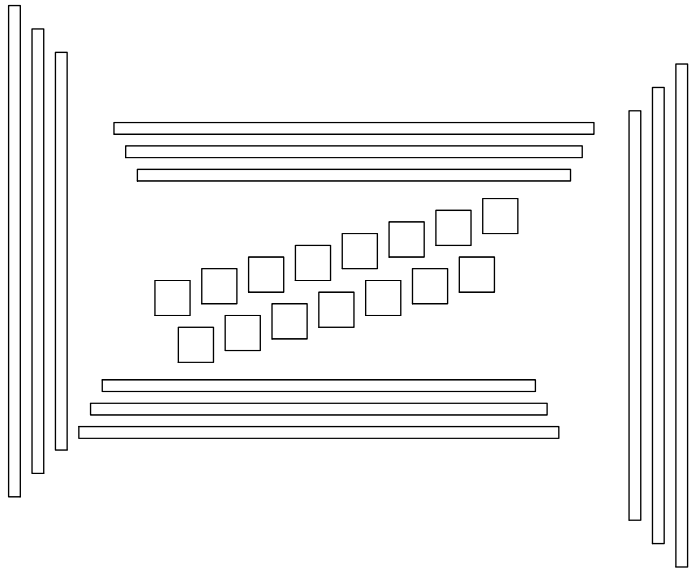

Figure 17: Construction of rectangle $k$-visibility hypergraph with $n$ vertices for $k = 2$, $n = 27$

Now, we construct $4k+4$ rectangles $A_i$, $B_i$, $C_i$, and $D_i$ for $1 \leq i \leq k+1$. Construct $k+1$ rectangles above these squares and $k+1$ rectangles below these squares. For $1 \leq i \leq k+1$, construct a rectangle $A_i$ with opposite vertices at $(-i, a+6+2i)$ and $(4a+4+(i+k+1), a+7+2i)$, and construct a rectangle $B_i$ with opposite vertices at $(-(i+k+1), -2i)$ and $(4a+4+i, -2i-1)$. Additionally, construct $k+1$ rectangles on the left and right sides. For $1 \leq i \leq k+1$, construct a rectangle $C_i$ with opposite vertices at $(-2k-1-2i, -2k-2-2i)$ and $(-2k-2-2i, a+9+4k+2i)$, and construct a rectangle $D_i$ with opposite vertices at $(4a+7+2k+2i, -4k-4-2i)$ and $(4a+8+2k+2i, a+8+2k+2i)$. This construction is shown in Figure 17 for $k = 2$ and $n = 27$.

We claim this rectangle $k$-visibility hypergraph has $(4k+6)n-4(k+1)(3k+5)$ edges. If we consider the vertical edges which contain the left side of each of the $n-4k-4$ squares, there are exactly $k+2$ edges for each side, so there are $(n-4k-4)(k+2)$ edges in this case. If we consider the vertical edges which are 0.1 to the right of the right edge of any square except the rightmost square, then for each square, there are $k+1$ such edges that would intersect the right edge of the square if it were moved 0.1 left. Thus, there are $(n-4k-5)(k+1)$ more edges. All of these edges are vertical, distinct, and pass through a square, and there are $(2k+3)n-(8k+13)(k+1)$ of them. Similarly, the number of horizontal edges that pass through a square is $(2k+3)n-(8k+13)(k+1)$, so the total number of edges passing through a square is $(4k+6)n-(16k+26)(k+1)$.

Now, we count the number of edges which only pass through rectangles. Any vertical edge only passes through the $A_i$ and $B_i$. The possible edges are $A_i$ through $A_{k+1}$, $B_{k+1}$ through $B_{k+2-i}$ for $1 \leq i \leq k+1$, $A_i$ through $A_{k+1}$, $B_j$ through $B_{j+1-i}$ for $1 \leq i \leq j \leq k$, and $B_i$ through $B_{k+1}$, $A_j$ through $A_{j+1-i}$ for $1 \leq i \leq j \leq k$. The total number of vertical edges is $(k+1)^2$.

Similarly, the number of horizontal edges which only pass through the $C_i$ and $D_i$ is $(k+1)^2$. There are additional horizontal edges which pass through exactly one of the $A_i$ or $B_i$. For each of these $2k+2$ rectangles, there are $k+2$ horizontal edges which intersect that rectangle and $k+1$ of the $C_i$ and $D_i$. Therefore, the total number of horizontal edges is $(k+1)^2 + 2(k+1)(k+2) = (k+1)(3k+5)$.

By adding up all of the cases, the number of edges in this rectangle $k$-visibility hypergraph is $(4k+6)n-(16k+26)(k+1)+(k+1)^2+(k+1)(3k+5) = (4k+6)n-4(k+1)(3k+5)$. □

## 6 Acknowledgements

I would like to thank my mentor Jesse Geneson for his invaluable support and guidance throughout my research. I would also like to thank the MIT PRIMES-USA program for providing me with this research opportunity and making this collaboration possible.

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