ON YETTER–DRINFELD MODULES OVER THE PANSERA ALGEBRAS

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Abstract. Generalizations of the non-commutative, non-cocommutative semisimple eight-dimensional Hopf algebra H_8 by Pansera in 2017 and Lomp in 2025 have found a great deal of interest. Classification of Hopf algebras with coradical not commutative or cocommutative is very difficult, making examples of classifications interesting. In 2016, Shi classified all finite-dimensional Hopf algebras with coradical the Kac-Paljutkin algebra H_8 using the lifting method, and of particular interest, found the positive part of $U_q(\mathfrak{sl}_3)$ at $q = -1$ as a Nichols algebra over H_8.

In this work, we present partial progress towards the goal of classifying finite-dimensional Hopf algebras over H_{2n^2} by computing some simple Yetter–Drinfeld modules over the Pansera algebras H_{2n^2}. In particular, we show that the category of comodules ${}^{H_{2n^2}}\mathcal{M}$ is Tambara–Yamagami. As a consequence, we see that all simple Yetter–Drinfeld modules are 1-dimensional, 2-dimensional, or n-dimensional, and compute the number of simple non-isomorphic Yetter–Drinfeld modules of each dimension. We classify and realize all simple non-isomorphic 1 and 2-dimensional Yetter–Drinfeld modules, and some of the simple n-dimensional Yetter–Drinfeld modules.

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1. Introduction

Hopf algebras are powerful mathematical objects that may be thought of as "infinitesimal quantized" symmetries, and were discovered by Heinz Hopf in the 1940s within the context of cohomology rings in algebraic topology [2]. They have since found applications in many fields. Some particularly interesting applications follow below.

Tensor categories are an analog of rings in the categorical setting, and have applications in fields like topological quantum field theory, representation theory of quantum groups, and more [6, 20]. Of particular interest to physicists, a modular tensor category furnishes a topological quantum field theory, or TQFT [17]. More generally, the category of representations of a finite-dimensional Hopf algebra furnishes a tensor category [7]. Thus, Hopf algebras can give rise to interesting tensor categories and related objects, such as TQFTs. This has practical applications: for example, Kitaev developed an error-resistant quantum computation system using Hopf algebras in 1997 [13].

From a mathematical perspective, groups can be philosophically thought of as Hopf algebras over the "field of one element." Thus, Hopf algebras can be thought of as a natural generalization of groups. The question of classifying all Hopf algebras up to isomorphism over an algebraically-closed field of characteristic zero, as posed by Kaplansky in 1975, is then of fundamental mathematical interest, for it generalizes the classical problem of classification of groups [11].

However, due to a lack of standard techniques, this problem is extremely difficult. In the case where the coradical H_0 is a fixed Hopf subalgebra, the lifting method provides a way to classify such Hopf algebras. This is analogous to classifying groups with a given maximal normal subgroup. The procedure, given in [3], is as follows:

Let H be a Hopf algebra such that the coradical H_0 is a Hopf subalgebra of H. The associated graded Hopf algebra of H is isomorphic to $R\#H_0$ where $R = \bigoplus_{n\in\mathbb{N}_0} R(n)$ is a braided Hopf algebra in the category of Yetter–Drinfeld modules over H_0 and $\#$ is the *Radford biproduct* or *bosonization* of R with H_0. To find all Hopf algebras H with coradical H_0, one proceeds as follows.

(1) Classify all Yetter–Drinfeld modules V over H_0.
(2) Find the Nichols algebras $\mathfrak{B}(V)$ of finite dimension; find an efficient set of relations for these algebras.
(3) If $R = \bigoplus_{n\in\mathbb{N}_0} R(n)$ is a finite-dimensional Hopf algebra in ${}^{H_0}_{H_0}\mathcal{YD}$ with $V = R(1)$, decide if $R \cong \mathfrak{B}(V)$. Here, $V = R(1)$ is the infinitesimal braiding.
(4) Given V as in (1), classify all H such that $\mathrm{gr}H \cong \mathfrak{B}(V)\#H_0$.

The lifting method was crucial for the classification of finite-dimensional pointed Hopf algebras and was applied to finite-dimensional copointed Hopf algebras [4, 5].

The Kac-Paljutkin algebra H_8 is, by dimension, the first semisimple non-commutative and non-cocommutative Hopf algebra. In 2017, Deividi Pansera constructed a generalization of the Kac-Paljutkin algebra, the Pansera algebra H_{2n^2}, as a C_2 extension of the group algebra $\Bbbk[C_n \times C_n]$ [15]. The classical Kac-Paljutkin algebra H_8 arises as the $n = 2$ case of the Pansera algebra. Recently, Christian Lomp constructed a further generalization of H_8 by a crossed product $H_{n,m} = \Bbbk Z_n^{\otimes m} \#_\gamma \sum_m$, with $H_{n,m}$ of dimension $n^m m!$ [14]. The Pansera algebra arises as the $m = 2$ case of Lomp's algebra.

In 2016, Yuxing Shi applied the lifting method to classify Hopf algebras with coradical H_8 [18]. Given the scarcity of classification results over algebras that are simultaneously non-commutative and non-cocommutative, alongside the wide interest in Pansera and Lomp's algebras, it is interesting to extend Shi's work to the Pansera algebras. The first step in this analysis is computing the Yetter–Drinfeld modules over the Pansera algebras H_{2n^2}.

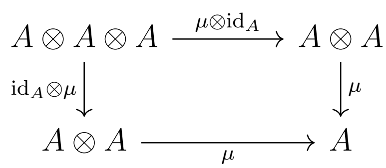
Of particular interest, in the case of H_8, Shi found that the positive part of $U_q(\mathfrak{sl}_3)$ at $q = -1$ appeared as a Nichols algebra over the Yetter–Drinfeld modules of H_8. In a sense, the Pansera algebras generalize H_8 by replacing $q = -1$ with q an nth primitive root of unity. Thus, when studying the Nichols algebras arising from Yetter–Drinfeld modules over H_{2n^2}, we expect to find the positive part of $U_q(\mathfrak{sl}_3)$ at an nth root of unity. Taking Drinfeld doubles, one would then find a Hopf algebra with a non-commutative Cartan part extending the structure of $U_q(\mathfrak{sl}_3)$ for q a primitive nth root of unity. As $U_q(\mathfrak{sl}_3)$ describes non-abelian anyons in condensed matter physics, this could give rise to interesting topological phases of matter and non-abelian quantum computing systems.

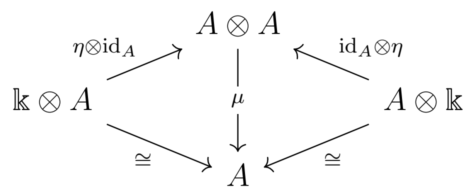
The paper is organized as follows. In Section 2, we review the basics of Hopf algebras and other relevant theory. In Section 3, we discuss the module structures arising from the induction method. In Section 4, we compute the category ${}^{H_{2n^2}}\mathcal{M}$ of left comodules over H_{2n^2}, in particular showing this category is a Tambara–Yamagami category. In Section 5, we show all Yetter–Drinfeld modules are 1-dimensional, 2-dimensional, or n-dimensional, and compute the number of Yetter–Drinfeld modules of each dimension. We then classify and realize all 1 and 2-dimensional Yetter–Drinfeld modules, and some n-dimensional Yetter–Drinfeld modules.

2. Background

2.1. **Hopf algebras and Yetter–Drinfeld modules.** Here, we review the basic theory of Hopf algebras and Yetter–Drinfeld modules, following [10]. Let $\Bbbk$ denote an algebraically closed field of characteristic zero, and assume all algebraic objects or operations are defined or taken over the field $\Bbbk$.

Definition 2.1 (Algebra)**.** Let A be a vector space, and $\mu : A \otimes A \to A$ and $\eta : \Bbbk \to A$ be maps called *multiplicaion* and *unit*, respectively. Then, (A, μ, η) is an *algebra* if the following diagrams commute:

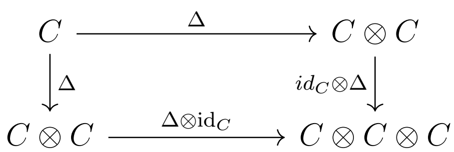


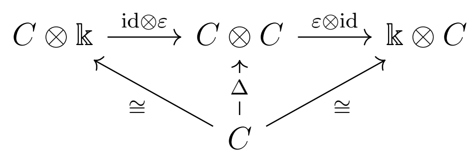


These diagrams are called associativity and unit diagrams, respectively. We sometimes say A is an algebra, omitting the maps μ and η when the context is clear.

We can obtain the object dual to an algebra by formally reversing the arrows.

Definition 2.2 (Coalgebra)**.** Let C be a vector space, and let $\Delta : C \to C \otimes C, \varepsilon : C \to \Bbbk$ be linear maps called *comultiplication* and *counit*, respectively. Then, (C, Δ, ε) is a *coalgebra* if the following diagrams commute:





These diagrams are called coassociativity and counit diagrams, respectively. We sometimes say C is a coalgebra, omitting the maps Δ and ε when the context is clear.

The Δ map in general sends an element $c \in C$ to an element $\Delta(c) = \sum_{i=1}^{n} c_{(1)i} \otimes c_{(2)i}$. Computations with Δ can thus become long and difficult to parse. This is mitigated by the following notation:

Remark 2.3 (Sweedler Notation)**.** We denote $\Delta(c) = \sum_{i=1}^{n} c_{(1)i} \otimes c_{2(i)}$ by $\Delta(c) = c_{(1)} \otimes c_{(2)}$, where implicit summation is understood. In this notation, we can compactly express the coassociativity and counit relations by:

$$\Delta(c_{(1)}) \otimes c_{(2)} = c_{(1)} \otimes \Delta(c_{(2)}), \qquad \varepsilon(c_{(1)})c_{(2)} = c = c_{(1)}\varepsilon(c_{(2)}).$$

The left equation encodes the coassociativity axiom, and the right the counit axiom.

Given a coalgebra, we may endow the dual vector space with an algebra structure:

Example 2.4 (Dual algebra)**.** If (C, Δ, ε) is a coalgebra, the dual space $C^* = \mathrm{Hom}(C, \Bbbk)$ becomes the *dual algebra* $(C^*, \mu_{C^*}, \eta_{C^*})$ with μ_{C^*}, η_{C^*} defined by:

$$\mu_{C^*} : C^* \otimes C^* \xrightarrow{\lambda_{C,C}} (C \otimes C)^* \xrightarrow{\Delta^*} C^*,$$

$$\eta_{C^*} : \Bbbk \cong \Bbbk^* \xrightarrow{\varepsilon^*} C^*,$$

where $\lambda_{C,C}$ is the standard injective homomorphism from $C^* \otimes C^*$ to $(C \otimes C)^*$ and ε^* is the linear functional dual to ε.

For finite-dimensional algebras, we may perform the reverse construction.

Example 2.5 (Dual coalgebra)**.** If (A, μ, η) is a finite-dimensional algebra, the dual space A^* becomes the *dual coalgebra* $(A^*, \Delta_{A^*}, \varepsilon_{A^*})$ with $\Delta_{A^*}, \varepsilon_{A^*}$ defined by:

$$\Delta_{A^*} : A^* \xrightarrow{\mu^*} (A \otimes A)^* \xrightarrow{(\lambda_{A,A})^{-1}} A^* \otimes A^*,$$
$$\varepsilon_{A^*} : \Bbbk \to \Bbbk^* \xrightarrow{\eta^*} A^*,$$

with μ^* the linear functional dual to μ and η^* the linear functional dual to η. Notice the map $(\lambda_{A,A})^{-1}$ exists since A is finite-dimensional.

Morphisms of algebras and coalgebras are as follows:

Definition 2.6 (Algebra homomorphisms)**.** Let A, B be algebras. An *algebra homomorphism* $\rho : A \to B$ is a linear map such that the following diagrams commute:

$$\begin{CD} A \otimes A @>{\rho\otimes\rho}>> B \otimes B \\ @V{\mu_A}VV @VV{\mu_B}V \\ A @>>{\rho}> B \end{CD} \qquad \begin{array}{ccc} A & \xrightarrow{\rho} & B \\ {\scriptstyle\eta_A}\nwarrow & & \nearrow{\scriptstyle\eta_B} \\ & \Bbbk & \end{array}$$

.

Definition 2.7 (Coalgebra homomorphisms)**.** Let C, D be coalgebras. A *coalgebra homomorphism* $\varphi : C \to D$ is a linear map such that the following diagrams commute:

$$\begin{CD} C @>{\varphi}>> D \\ @V{\Delta_C}VV @VV{\Delta_D}V \\ C \otimes C @>>{\varphi\otimes\varphi}> D \otimes D \end{CD} \qquad \begin{array}{ccc} C & \xrightarrow{\varphi} & D \\ {\scriptstyle\varepsilon_C}\searrow & & \nearrow{\scriptstyle\varepsilon_D} \\ & \Bbbk & \end{array}$$

.

We now recall the definition of modules over an algebra and the corresponding notion for coalgebras.

Definition 2.8 (Module)**.** A (left) *module* (V, ρ) over an algebra (A, μ, η) is a vector space V equipped with an *action* $\rho : A \otimes V \to V$ such that the following diagrams commute:

$$\begin{CD} A \otimes A \otimes V @>{\mu\otimes\mathrm{id}_V}>> A \otimes V \\ @V{\mathrm{id}_A\otimes\rho}VV @VV{\rho}V \\ A \otimes V @>>{\rho}> V \end{CD} \qquad \begin{array}{ccc} \Bbbk \otimes V & \xrightarrow{\eta\otimes\mathrm{id}_V} & A \otimes V \\ & \underset{\cong}{\searrow} & \Big\downarrow{\scriptstyle\rho} \\ & & V \end{array} .$$

Dually, for coalgebras, we reverse arrows.

Definition 2.9 (Comodule)**.** A (left) *comodule* (V, δ) over a coalgebra (C, Δ, ε) is a vector space V equipped with a *coaction* $\delta : V \to C \otimes V$ such that the following diagrams commute:

$$\begin{array}{ccc} V & \xrightarrow{\quad\delta\quad} & C\otimes V \\ {\scriptstyle\delta}\big\downarrow & & \big\downarrow{\scriptstyle\Delta\otimes\mathrm{id}_V} \\ C\otimes V & \xrightarrow[\mathrm{id}_C\otimes\delta]{} & C\otimes C\otimes V \end{array} \qquad \begin{array}{ccc} V & \xrightarrow{\delta} & C\otimes V \\ & {\scriptstyle\cong}\searrow & \big\downarrow{\scriptstyle\varepsilon\otimes\mathrm{id}_V} \\ & & \Bbbk\otimes V \end{array}$$

When an object is both an algebra and a coalgebra, a compatibility condition may be imposed.

Definition 2.10 (Bialgebra)**.** Let B be a vector space such that (B,μ,η) is an algebra and (B,Δ,ε) is a coalgebra. Then, $(B,\mu,\eta,\Delta,\varepsilon)$ is a *bialgebra* if Δ and ε are algebra morphisms with respect to (B,μ,η).

Products of comodules over a bialgebra are given by the following:

Definition 2.11 (Tensor product of comodules over a bialgebra)**.** Given a bialgebra B and V,W left B-comodules with coactions $\delta_V : V \to B\otimes V$ by $\delta_V(v) = v_{(-1)}\otimes v_{(0)}$ and $\delta_W : W \to B\otimes W$ by $\delta_W(w) = w_{(-1)}\otimes w_{(0)}$ respectively, $V\otimes W$ may be endowed with a comodule structure $(V\otimes W, \delta_{V\otimes W})$ where $\delta_{V\otimes W}$ is given by:

$$\delta_{V\otimes W}(v\otimes w) = (\mu\otimes \mathrm{id}_{V\otimes W})(v_{(-1)}\otimes w_{(-1)}\otimes v_{(0)}\otimes w_{(0)}),$$

for all $v\in V$ and $w\in W$.

To turn a bialgebra into a Hopf algebra, we impose a map similar to an inverse.

Definition 2.12 (Hopf Algebra)**.** Consider $(H,\mu,\eta,\Delta,\varepsilon)$ a bialgebra. We say that H is a Hopf algebra if there exists a linear map $S : H\to H$ such that, for all $h\in H$, $h_{(1)}S(h_{(2)}) = \varepsilon(h)1 = S(h_{(1)})h_{(2)}$. Equivalently, the following diagrams must commute:

$$\begin{array}{ccc} H & \xrightarrow{\quad\Delta\quad} & H\otimes H \\ {\scriptstyle\eta\circ\varepsilon}\big\downarrow & & \big\downarrow{\scriptstyle S\otimes\mathrm{id}_H} \\ H & \xleftarrow[\mu]{} & H\otimes H \end{array} \qquad \begin{array}{ccc} H & \xrightarrow{\quad\Delta\quad} & H\otimes H \\ {\scriptstyle\eta\circ\varepsilon}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{id}_H\otimes S} \\ H & \xleftarrow[\mu]{} & H\otimes H \end{array}$$

We call S the *antipode* of H.

The objects we wish to study are the Yetter–Drinfeld modules.

Definition 2.13 (Yetter–Drinfeld modules)**.** Given a Hopf algebra H, let V be a left H-module and a left H-comodule with left action and left coaction:

$$\rho : H\otimes V\to V, \qquad h\otimes v \xrightarrow{\rho} h\cdot v = hv,$$

$$\delta : V\to H\otimes V, \qquad v \xrightarrow{\delta} v_{(-1)}\otimes v_{(0)},$$

Then (V,ρ,δ) is a (left)-(left) *Yetter–Drinfeld module* over H if for all $h\in H, v\in V$:

$$\delta(h\cdot v) = h_{(1)}v_{(-1)}S(h_{(3)})\otimes h_{(2)}\cdot v_{(0)}.$$

We denote the category of left-left Yetter–Drinfeld modules over H by ${}^H_H\mathcal{YD}$.

Yetter–Drinfeld modules encode combinatorial information via a *braiding*. This is not central to the paper at hand, but is important for the general theory of Nichols algebras, which is relevant for the Andruskiewitsch–Schneider lifting method.

2.2. **Some categorical notions.** Here, we introduce some fundamental notions from category theory.

Recall the definition of a Tambara–Yamagami category from [19].

Definition 2.14. Let A be a finite abelian group and $\Bbbk$ an algebraically closed field of characteristic 0. A *Tambara–Yamagami category* associated to A is a tensor category over $\Bbbk$ with finitely-many simple objects

$$\{U_a\}_{a\in A} \coprod \{U_m\},$$

such that U_a is invertible while U_m is noninvertible, with the fusion rules on simple objects given by:

$$\begin{aligned} U_a \otimes U_{a'} &= U_{aa'}, \\ U_a \otimes U_m &= U_m, \\ U_m \otimes U_a &= U_m, \\ U_m \otimes U_m &= \bigoplus_{a''\in A} U_{a''}, \end{aligned}$$

for all $a, a' \in A$.

Remark 2.15. Such Tambara-Yamagami categories can be shown to be fusion categories, and are classified up to equivalence, by a symmetric bicharacter

$$\chi : A \times A \to \Bbbk^\times$$

together with a scalar $\tau \in \Bbbk^\times$ satisfying $\tau^2 = |A|^{-1}$ [19].

For the following, we follow Chapter 3 of [7].

Definition 2.16 (Unital $\mathbb{N}$-ring)**.** A unital $\mathbb{N}$-rind is a ring R such that:

(1) the underlying abelian group is free abelian
(2) there exists a finite $\mathbb{N}$-basis: a finite set $I \subset \mathbb{N}$ of elements $X_i \in R$, $i \in I$, such that:

$$X_i X_j = \sum_{k\in I} c_{ij}^k X_k,$$

for $c_{ij}^k \in \mathbb{N}$.
(3) the ring unit 1 is one of these X_i.

Recall the following foundational result:

Theorem 2.17 (Frobenius-Perron)**.** *Let B be a square matrix with non-negative real entries. Then B has a non-negative real eigenvalue.*

For a unital $\mathbb{N}$-ring, we may define the *Frobenius-Perron* dimension:

Definition 2.18 (Frobenius-Perron Dimension)**.** Let R be a unital $\mathbb{N}$-ring with finite $\mathbb{N}$-basis I. Then, for each X in R, we may define an $|I|\times|I|$ matrix of left multiplication, that is, a matrix $(N_X)_{ij}$ defined by:

$$XX_j = \sum_{i\in I}(N_x)_{ij}X_i.$$

The largest eigenvector of the matrix N_X is the *Frobenius-Perron dimension* of X.

Remark 2.19. By Theorem 2.17, the *Frobenius-Perron* dimension of any object in a unital $\mathbb{N}$-ring is always a non-negative real number. Furthermore, the eigenvalue may be considered as the root of the characteristic polynomial of the matrix N_x, which is integer-valued. Thus, the eigenvalue will be a non-negative algebraic integer.

Remark 2.20. For R a ring of representations of a Hopf algebra, the Frobenius-Perron dimension coincides with the vector space dimension. See Chapter 6 of [7] for a proof.

2.3. **The Drinfeld double.** Here, we define the Drinfeld double of a Hopf algebra.

Definition 2.21 (Drinfeld double)**.** Let $(H,\mu,\eta,\Delta,\varepsilon,S_H)$ be a finite-dimensional Hopf algebra such that the antipode S_H is bijective. Let $D(H)=(H^*)^{\text{op, cop}}\otimes H$, and construct the maps μ_D,η_D,Δ_D, ε_D, and S_D by:

$$\begin{aligned}\mu_D(f\otimes x, g\otimes y) &= fg\left(S^{-1}(x_{(3)})\cdot ?\cdot x_{(1)}\right)\otimes x_{(2)}y\\ \Delta_D(f\otimes x) &= \left(f_{(2)}\otimes x_{(1)}\right)\otimes\left(f_{(1)}\otimes x_{(2)}\right)\\ \eta_D(k) &= \varepsilon\otimes\eta_H(k)\\ \varepsilon_D(f\otimes x) &= \varepsilon_{H^*}(f)\cdot\varepsilon_H(x),\end{aligned}$$

and

$$S_D(f\otimes x) = (\varepsilon\otimes S_H(x))\cdot_D\left(S_{H^*}^{-1}(f)\otimes\eta_H(1)\right),$$

for all $f,g\in H^*$, $x,y\in H$, and $g\left(S^{-1}(x_{(3)})\cdot ?\cdot x_{(1)}\right)$ is the element of H^* sending h to $g\left(S^{-1}(x_{(3)})\cdot h\cdot x_{(1)}\right)$.

Then $(D(H),\mu_D,\eta_D,\Delta_D,\varepsilon_D,S_D)$ is a Hopf algebra and is called the *Drinfeld double* of H.

Remark 2.22. This definition follows from section IX.4.1 of [12].

The following foundational theorem is an immediate corollary of Remark 4.1.5 in [10]:

Corollary 2.23. *Given a finite-dimensional Hopf algebra H with bijective antipode, there is an isomorphism of monoidal categories between ${}_{D(H)}\mathcal{M}$ and ${}^H_H\mathcal{YD}$.*

2.4. **The lifting method.** The following lemma, proved in a different form as Proposition 2 in [16], will allow us to determine many simple Yetter-Drinfeld modules.

Lemma 2.24 (Radford)**.** *Let H be a Hopf algebra over the field $\Bbbk$ with antipode S.*

(1) If L is a simple (left) H-module, then $L \otimes H \in {}^H_H\mathcal{YD}$, the module and comodule actions are given by:

$$h \cdot (l \otimes a) = h_{(2)} \cdot l \otimes h_{(3)} a S^{-1}(h_{(1)}), \qquad \delta(l \otimes h) = h_{(1)} \otimes (l \otimes h_{(2)}),$$

for every $h, a \in H, l \in L$.

(2) Any simple (left-left) Yetter–Drinfeld module $M \in {}^H_H\mathcal{YD}$ is isomorphic to a Yetter–Drinfeld submodule of some $L \otimes H$ as described above.

2.5. **The Pansera algebra H_{2n^2}.** Here, we describe the Pansera algebra H_{2n^2}.

Definition 2.25. Let $n > 1$ and q be a primitive n-th root of unity. The Pansera algebra H_{2n^2} is the associative Hopf algebra generated by x, y, z, with the following relations:

$$x^n = 1, \quad y^n = 1, \quad z^2 = \frac{1}{n} \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} q^{-ij} x^i y^j,$$

$$xy = yx, \quad zx = yz, \quad zy = xz.$$

The comultiplication is determined by:

$$\Delta(x) = x \otimes x, \quad \Delta(y) = y \otimes y, \quad \Delta(z) = \frac{1}{n} \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} q^{-ij} (x^i z) \otimes (y^j z).$$

The counit is determined by $\varepsilon(x) = \varepsilon(y) = \varepsilon(z) = 1$. The antipode is given by $S(x) = x^{-1}, S(y) = y^{-1}, S(z) = z$. The Pansera algebra H_{2n^2} is $2n^2$-dimensional with basis

$$\{x^i y^j, x^i y^j z \mid 0 \leq i, j \leq n-1\}.$$

The following elementary lemma is key to the remainder of the paper.

Lemma 2.26. *The following is true:*

$$\sum_{r=0}^{n-1} q^{rm} = \begin{cases} n, & \text{if } m \equiv 0 \pmod{n} \\ 0, & \text{otherwise.} \end{cases}$$

The following change of basis will be used throughout the paper:

Proposition 2.27. *Define:*

$$|r, s\rangle = \sum_{a,b=0}^{n-1} q^{-ra} q^{-sb} x^a y^b, \qquad |r, s\rangle_z = \sum_{a,b=0}^{n-1} q^{-ra} q^{-sb} x^a y^b z.$$

Then, $\{|r, s\rangle, |r, s\rangle_z \mid r, s \in \{0, \ldots, n-1\}\}$ is a basis for H_{2n^2}.

Proof. It suffices to find a linear inverse map between the bases $x^a y^b$ and $|r,s\rangle$. Applying Lemma 2.26, we see the following expansion:

$$x^a y^b = \frac{1}{n^2}\sum_{r,s=0}^{n-1} q^{ra}q^{sb}\left(\sum_{c,d} q^{-rc}q^{-rd}x^c y^d\right) = \frac{1}{n^2}\sum_{r,s=0}^{n-1} q^{ra}q^{sb}\,|r,s\rangle\,.$$

The same calculation shows the result for $x^a y^b z$ and $|r,s\rangle_z$. □

The modules over H_{2n^2} were classified in [8].

Proposition 2.28 (Ferraro et al.). *Let p be a square root of q, or equivalently, a $2n$th primitive root of unity. Then the simple modules over H_{2n^2} are given by:*

(i) $T_k^{\pm} = \Bbbk\{v_k^{\pm}\}$ *with:*

$$x\cdot v_k^{\pm} = q^k v_k^{\pm} \qquad y\cdot v_k^{\pm} = q^k v_k^{\pm} \qquad z\cdot v_k^{\pm} = \pm p^{k^2} v_k^{\pm},$$

for $k\in\mathbb{Z}_n$.

(ii) $\pi(i,j) = \Bbbk\{v_1, v_2\}$ *with* $v_1 = \begin{pmatrix}1\\0\end{pmatrix}, v_2 = \begin{pmatrix}0\\1\end{pmatrix}$ *such that:*

$$x \to \begin{pmatrix} q^i & 0\\ 0 & q^j\end{pmatrix} \qquad y \to \begin{pmatrix} q^j & 0\\ 0 & q^i\end{pmatrix} \qquad z \to \begin{pmatrix} 0 & 1\\ q^{ij} & 0\end{pmatrix},$$

for $(i,j)\in\mathbb{Z}_n\times\mathbb{Z}_n$ *with* $i<j$.

In what follows, we will assume, unless otherwise stated, that all sums (in all indices) run from 0 to $n-1$, q is a primitive n-th root of unity, and p is a square root of q.

3. Induced Module Structure on H_{2n^2}

Recall from Lemma 2.24 that given a Hopf algebra H and a (left) H-module L, the induced (left) module structure on $L\otimes H$ is given by:

$$h\cdot(l\otimes a) = h_{(2)}\cdot l\otimes h_{(3)}aS^{-1}(h_{(1)}).$$

for all $a,h\in H$ and $l\in L$. Also recall $\Delta^2(x) = x\otimes x\otimes x$, $\Delta^2(y) = y\otimes y\otimes y$, and:

$$\Delta^2(z) = \frac{1}{n^2}\sum_{i,j,c,d} q^{-ij-cd}(x^{i+c}z)\otimes(x^i y^d z)\otimes(y^j z).$$

3.1. Induced module structure on $\mathbf{T_k^+}\otimes\mathbf{H_{2n^2}}$.

Let L be the one-dimensional simple (left) module T_k^+ given in (i) of Proposition 2.28. We consider the action of H_{2n^2} on basis elements of the form $t_k^+\otimes x^a y^b$ and $t_k^+\otimes x^a y^b z$. We have:

$$x\cdot(t_k^+\otimes x^a y^b) = (x\cdot t_k^+)\otimes(x\cdot(x^a y^b)S^{-1}(x)) = q^k(t_k^+\otimes x^a y^b),$$
$$y\cdot(t_k^+\otimes x^a y^b) = (y\cdot t_k^+)\otimes(y\cdot(x^a y^b)S^{-1}(y)) = q^k(t_k^+\otimes x^a y^b)$$

We compute $z \cdot (t_k^+ \otimes x^a y^b)$ here for illustrative purposes. We have:

$$z \cdot (t_k^+ \otimes x^a y^b) = \frac{1}{n^2} \sum_{i,j,c,d} q^{-ij-cd} \left(x^i y^d z \cdot t_k^+\right) \otimes \left(x^{i+c} z x^a y^b S^{-1}(y^j z)\right),$$
$$= \frac{p^{k^2}}{n^3} \sum_{i,j,c,d,u,v} q^{-ij-cd-uv+ik+dk} \left(t_k^+ \otimes x^{b+u+(i+c)} y^{a+v-j}\right).$$

Notice d is not part of the vector term $t_k^+ \otimes x^{b+u+(i+c)} y^{a+v-j}$, so we can rewrite the above:

$$z \cdot (t_k^+ \otimes x^a y^b) = \frac{p^{k^2}}{n^3} \sum_{i,j,c,u,v} \left(\sum_d q^{-cd+dk}\right) \left(q^{-ij-uv+ik} \left(t_k^+ \otimes x^{b+u+(i+c)} y^{a+v-j}\right)\right).$$

Now, using Lemma 2.26, the term $\sum_d q^{-cd+dk} = \sum_d q^{d(k-c)}$ is $n\delta_{0,k-c} = n\delta_{k,c}$. Evaluating the sum over c using the Kronecker delta, we get:

$$z \cdot (t_k^+ \otimes x^a y^b) = \frac{p^{k^2}}{n^2} \sum_{i,j,u,v} q^{-ij-uv+ik} \left(t_k^+ \otimes x^{b+u+(i+k)} y^{a+v-j}\right).$$

Substitute $u' = u + i + k$ and $v' = v - j$. Since the indices u, v range from 0 to $n-1$ and q, x, y are all of order n, u, v may be considered modulo n. This means summing over u' gives the same expression as summing over u. So we may replace u, v with u', v' in the sum indices to obtain:

$$z \cdot (t_k^+ \otimes x^a y^b) = \frac{p^{k^2}}{n^2} \sum_{i,j,u',v'} q^{-ij-(u'-i-k)(v'+j)+ik} \left(t_k^+ \otimes x^{b+u'} y^{a+v'}\right)$$
$$= \frac{p^{k^2}}{n^2} \sum_{u',v',j} \left(\sum_i q^{-ij+iv'+ij+ik}\right) \left(q^{-u'v'-u'j+kv'+kj} \left(t_k^+ \otimes x^{b+u'} y^{a+v'}\right)\right),$$

where we have been able to isolate the i-sum since the exponent no longer depends on i. By Lemma 2.26, we obtain a factor $n\delta_{v'+k,0}$. Cancelling with the sum over v' gives:

$$z \cdot (t_k^+ \otimes x^a y^b) = \frac{p^{k^2}}{n} \sum_{j,u'} q^{-u'(-k)-u'j+k(-k)+kj} \left(t_k^+ \otimes x^{b+u'} y^{a-k}\right).$$

Applying Lemma 2.26 to $\sum_j q^{-u'j+kj}$ gives:

$$z \cdot (t_k^+ \otimes x^a y^b) = p^{k^2} q^{k(k)-k^2} (t_k^+ \otimes x^{b+k} y^{a-k}) = p^{k^2} (t_k^+ \otimes x^{b+k} y^{a-k}).$$

We omit similar calculations in the induced modules for brevity, finding:

Lemma 3.1. *The induced module action on $T_k^+ \otimes H_{2n^2}$ is given by:*

$$x \cdot (t_k^+ \otimes x^a y^b) = q^k \left(t_k^+ \otimes x^a y^b\right), \qquad x \cdot (t_k^+ \otimes x^a y^b z) = q^k \left(t_k^+ \otimes x^{a+1} y^{b-1} z\right),$$

$$y \cdot (t_k^+ \otimes x^a y^b) = q^k \left(t_k^+ \otimes x^a y^b\right), \qquad y \cdot (t_k^+ \otimes x^a y^b z) = q^k \left(t_k^+ \otimes x^{a-1} y^{b+1} z\right),$$

$$z \cdot (t_k^+ \otimes x^a y^b) = p^{k^2} \left(t_k^+ \otimes x^{b+k} y^{a-k}\right),$$

$$z \cdot (t_k^+ \otimes x^a y^b z) = \frac{p^{k^2}}{n} \sum_{u,v} q^{-uv-v^2}\, t_k^+ \otimes x^{b+u} y^{a+v}.$$

3.2. **Induced module structure on $\mathbf{T_k^-} \otimes \mathbf{H_{2n^2}}$.** Let L be the one-dimensional simple (left) module T_k^- given in (i) of Proposition 2.28. We consider the action of H_{2n^2} on basis elements of the form $t_k^- \otimes x^a y^b$ and $t_k^- \otimes x^a y^b z$. We obtain:

Lemma 3.2. *The induced module action on $T_k^- \otimes H_{2n^2}$ is given by:*

$$x \cdot (t_k^- \otimes x^a y^b) = q^k \left(t_k^- \otimes x^a y^b\right), \qquad x \cdot (t_k^- \otimes x^a y^b z) = q^k \left(t_k^- \otimes x^{a+1} y^{b-1} z\right),$$

$$y \cdot (t_k^- \otimes x^a y^b) = q^k \left(t_k^- \otimes x^a y^b\right), \qquad y \cdot (t_k^- \otimes x^a y^b z) = q^k \left(t_k^- \otimes x^{a-1} y^{b+1} z\right),$$

$$z \cdot (t_k^- \otimes x^a y^b) = -p^{k^2} \left(t_k^- \otimes x^{b+k} y^{a-k}\right),$$

$$z \cdot (t_k^- \otimes x^a y^b z) = -\frac{p^{k^2}}{n} \sum_{u,v} q^{-uv-v^2}\, t_k^- \otimes x^{b+u} y^{a+v}.$$

3.3. **Induced module structure on** $\pi(i,j) \otimes H_{2n^2}$**.** Let L be the two-dimensional module $\pi(i,j)$ given in (ii) of Proposition 2.28. We consider the action of H_{2n^2} on basis elements of the form $v_1 \otimes x^a y^b$, $v_2 \otimes x^a y^b$, $v_1 \otimes x^a y^b z$, $v_2 \otimes x^a y^b z$. We find:

Lemma 3.3. *The induced module structure on $\pi(i,j) \otimes H_{2n^2}$ is given by:*

$$x \cdot \left(v_1 \otimes x^a y^b\right) = q^i \left(v_1 \otimes x^a y^b\right), \qquad x \cdot \left(v_2 \otimes x^a y^b\right) = q^j \left(v_2 \otimes x^a y^b\right),$$

$$x \cdot \left(v_1 \otimes x^a y^b z\right) = q^i \left(v_1 \otimes x^{a+1} y^{b-1} z\right), \qquad x \cdot \left(v_2 \otimes x^a y^b z\right) = q^j \left(v_2 \otimes x^{a+1} y^{b-1} z\right),$$

$$y \cdot \left(v_1 \otimes x^a y^b\right) = q^j \left(v_1 \otimes x^a y^b\right), \qquad y \cdot \left(v_2 \otimes x^a y^b\right) = q^i \left(v_2 \otimes x^a y^b\right),$$

$$y \cdot \left(v_1 \otimes x^a y^b z\right) = q^j \left(v_1 \otimes x^{a-1} y^{b+1} z\right), \qquad y \cdot \left(v_2 \otimes x^a y^b z\right) = q^i \left(v_2 \otimes x^{a-1} y^{b+1} z\right),$$

$$z \cdot \left(v_1 \otimes x^a y^b\right) = q^{ij} \left(v_2 \otimes x^{b+i} y^{a-j}\right), \qquad z \cdot \left(v_2 \otimes x^a y^b\right) = v_1 \otimes x^{b+j} y^{a-i},$$

$$z \cdot \left(v_1 \otimes x^a y^b z\right) = \frac{q^{ij}}{n} \sum_{u,v} q^{-uv-v^2+v(i-j)} v_2 \otimes x^{b+u} y^{a+v} z,$$

$$z \cdot \left(v_2 \otimes x^a y^b z\right) = \frac{1}{n} \sum_{u,v} q^{-uv-v^2+v(j-i)} v_1 \otimes x^{b+u} y^{a+v} z.$$

4. Comodules over H_{2n^2}

Here, we discuss comodules over H_{2n^2}. We classify simple objects of the category ${}^{H_{2n^2}}\mathcal{M}$ of (left) comodules over H_{2n^2} and show this category is Tambara–Yamagami.

4.1. **One-dimensional comodules.** We have the following classification:

Proposition 4.1. *There are n^2 one-dimensional, non-isomorphic. simple comodules $(W_{(a,b)}, \delta)$ over H_{2n^2}, with $W_{(a,b)} = \Bbbk\{w_{(a,b)}\}$ and coaction given by:*

$$\delta(w_{(a,b)}) = x^a y^b \otimes w_{(a,b)},$$

for $(a, b) \in \mathbb{Z}_n^2$.

Proof. To check coassociativity, we compute:

$$(\mathrm{id}_{H_{2n^2}} \otimes \delta) \circ \delta(w_{(a,b)}) = (\mathrm{id}_{H_{2n^2}} \otimes \delta)(x^a y^b \otimes w_{(a,b)}) = x^a y^b \otimes x^a y^b \otimes w_{(a,b)},$$
$$(\Delta \otimes \mathrm{id}_{w_{(a,b)}}) \circ \delta(w_{(a,b)}) = (\Delta \otimes \mathrm{id}_{w_{(a,b)}})(x^a y^b \otimes w_{(a,b)}) = x^a y^b \otimes x^a y^b \otimes w_{(a,b)}.$$

The counit is computed similarly, as:

$$(\varepsilon \otimes \mathrm{id}) \circ \delta(w_{(a,b)}) = \varepsilon(x^a y^b) \otimes w_{(a,b)} = 1 \otimes w_{(a,b)}.$$

Clearly, for $(a, b) \neq (c, d)$, $W_{(a,b)}$ and $W_{(c,d)}$ are non-isomorphic as comodules. □

4.2. **On the dual algebra $\mathbf{H_{2n^2}}^*$.** We will show that the only remaining comodule is a unique simple comodule of dimension n. To find this, recall the following well-known equivalence of categories, which may be seen in the discussion immediately following Theorem 2.1 in [1]:

Proposition 4.2. *Let C be a finite-dimensional coalgebra. Then there are mutually inverse functors*

$$F : {}^C\mathcal{M} \longrightarrow {}_{C^*}\mathcal{M} \quad \text{and} \quad G : {}_{C^*}\mathcal{M} \longrightarrow {}^C\mathcal{M}.$$

The functor F assigns to a left C-comodule (V, δ) the left C^*-module (V, ρ_δ) defined by

$$\rho_\delta(f \otimes v) = (f \otimes \mathrm{id})\big(\delta(v)\big) = f(v_{-1})\, v_0,$$

for all $f \in C^*$ and $v \in V$.

Conversely, the functor G assigns to a left C^*-module (V, ρ) the left C-comodule (V, δ_ρ) given by

$$\delta_\rho(v) = \sum_i c_i \otimes \rho(c_i^* \otimes v),$$

where $\{c_i\}$ is a basis of C and $\{c_i^*\}$ is the corresponding dual basis of C^*.

Thus, to determine the remaining simple comodules of H_{2n^2}, it suffices to determine a simple n-dimensional (right) $H_{2n^2}^*$-module. We now find the multiplication in $H_{2n^2}^*$.

Denote the dual basis element corresponding to $x^a y^b$ by $f_{a,b}$ and the dual element of $x^a y^b z$ by $f_{a,b}^z$. Also, define a dual Fourier basis for the dual algebra $H_{2n^2}^*$ by:

$$g_{r,s} = \sum_{a,b} q^{ra} q^{sb} f_{a,b}, \qquad g_{r,s}^z = \sum_{a,b} q^{ra} q^{sb} f_{a,b}^z.$$

This is seen to be another basis by the same calculation as the Fourier basis for H_{2n^2}.

Notice:

$$\begin{aligned} g^z_{r,s}\left(|u,v\rangle_z\right) &= \sum_{a,b,c,d} q^{ra+sb-uc-vd} f^z_{a,b}(x^c y^d z) \\ &= \sum_{a,b,c,d} q^{ra+sb-uc-vd} \delta_{a,c}\delta_{b,d} \\ &= \left(\sum_a q^{a(r-u)}\right)\left(\sum_b q^{b(s-v)}\right) = n^2 \delta_{r,u}\delta_{s,v}. \end{aligned}$$

So $\frac{1}{n^2} g^z_{r,s}$ is the dual function to $|r,s\rangle_z$. For our purposes, it suffices only to compute the multiplication in the n^2-dimensional subalgebra $C' = \Bbbk\{g^z_{a,b}\}$. We have:

$$\begin{aligned} \mu(f^z_{a,b}, f^z_{c,d})(x^u y^v z) &= \left(f^z_{a,b} \otimes f^z_{c,d}\right) \circ \left(\Delta(x^u y^v z)\right) \\ &= \left(f^z_{a,b} \otimes f^z_{c,d}\right)\left(\frac{1}{n}\sum_{i,j} q^{-ij} x^{i+u} y^v z \otimes x^u y^{j+v} z\right) \\ &= \frac{1}{n}\sum_{i,j} q^{-ij} f^z_{a,b}(x^{i+u} y^v z) \otimes f^z_{c,d}(x^u y^{j+v} z) \\ &= \frac{1}{n}\sum_{i,j} q^{-ij} \delta_{a,i+u}\delta_{b,v}\delta_{c,u}\delta_{d,j+v}. \end{aligned}$$

The Kronecker delta conditions imply $u = c, v = b$ (if they are not zero), so we can rewrite the other Kronecker deltas as $\delta_{a,i+c}$ and $\delta_{d,j+b}$. Then these may be cancelled with the sums over i, j to force $i = a - c, j = d - b$, so substituting, we see:

$$\mu(f^z_{a,b}, f^z_{c,d})(x^u y^v z) = \frac{1}{n} q^{(a-u)(d-v)} \delta_{b,v}\delta_{c,u}.$$

But then $\delta_{b,v}\delta_{c,u}$ is exactly $f^z_{c,b}(x^u y^v z)$, so we see as a function:

$$\mu(f^z_{a,b}, f^z_{c,d}) = \frac{1}{n} q^{-(a-c)(b-d)} f^z_{c,b}.$$

Similarly, one can check the multiplication in the dual Fourier basis to find:

$$\mu(g^z_{r,s} \otimes g^z_{u,v}) = q^{rv} g^z_{r+u,s+v}.$$

Thus, $C' = \Bbbk\{f^z_{a,b}\} = \Bbbk\{g^z_{a,b}\}$ is a closed $H^*_{2n^2}$-subalgebra. Abstractly, we have:

Proposition 4.3. *Let $K = \Bbbk\langle a,b\rangle / \langle a^n = 1, b^n = 1, ab = qba\rangle$. Then, C' and K are isomorphic as algebras.*

Proof. Define a map $\psi : C' \to K$ by $g^z_{1,0} \to a$, $g^z_{0,1} \to b$. The multiplication rules give $g^z_{r,s} = q^{-rs}\left(g^z_{1,0}\right)^r\left(g^z_{0,1}\right)^s$. Thus, ψ is a bijection of generators by $g^z_{r,s} \longleftrightarrow q^{-rs} a^r b^s$. So, to show ψ is an isomorphism, it suffices to check that ψ is an algebra homomorphism, which is straightforward. □

We now identify an n-dimensional simple right K-module.

Proposition 4.4. *Let V_n be an n-dimensional vector space with basis $\{v_0, \ldots, v_{n-1}\}$ and define a right action of K by $\rho(v_i \otimes a) = v_{i+1 \pmod n}$ and $\rho(v_i \otimes b) = q^i v_i$. Then V_n is a simple right K-module.*

Proof. It is straightforward to check that V_n is a module. Suppose V' is any irreducible submodule. Then V' contains an eigenvector of $B : v \to \rho(v \otimes b)$ since $\Bbbk$ is algebraically closed. But these are precisely the v_i, so V' contains some v_i. Then, applying A n times to v_i gives all the v_j and thus all of V. Thus, V is an irreducible module. □

4.3. **An n-dimensional comodule.** Notice $g^z_{r,s} = q^{-rs} \left(g^z_{1,0}\right)^r \left(g^z_{0,1}\right)^s$. The linear functional $|r, s\rangle^*$ dual to $|r, s\rangle$ (resp. $|r, s\rangle^*_z$) may be written in our dual Fourier basis as $\frac{1}{n^2} g_{r,s}$ (resp. $\frac{1}{n^2} g^z_{r,s}$). Using the isomorphism from Proposition 4.3, it is straightforward to see the following:

Proposition 4.5. *There exists an n-dimensional simple module (V_n, ρ) over ${H_{2n^2}}^*$ given by $V_n = \Bbbk\{v_0, \ldots, v_{n-1}\}$ with action ρ defined by $\rho(v_i \otimes |r, s\rangle^*) = 0$ and $\rho(v_i \otimes |r, s\rangle^*_z) = q^{si} v_{i+r \pmod n}$.*

To find an n-dimensional simple (left) comodule over H_{2n^2}, we apply the inverse functor $G : {}_{{H_{2n^2}}^*}\mathcal{M} \to {}^{H_{2n^2}}\mathcal{M}$. Thus we recover an n-dimensional simple comodule over H_{2n^2} by $V_n = \Bbbk\{v_0, \ldots, v_{n-1}\}$ by:

$$\begin{aligned}
\delta(v_j) &= \sum_{c_i \text{ a basis}} c_i \otimes (\rho(v_j \otimes c_i^*)) \\
&= \left(\sum_{a,b} |a, b\rangle \otimes \rho \left(v_j \otimes \frac{1}{n^2} g_{a,b} \right) \right) + \left(\sum_{r,s} |r, s\rangle_z \otimes \rho \left(v_j \otimes \frac{1}{n^2} g^z_{r,s} \right) \right) \\
&= \frac{1}{n^2} \sum_{r,s} q^{sj} |r, s\rangle_z \otimes v_{j+r}.
\end{aligned}$$

So, we finally recover the following proposition:

Proposition 4.6. *There is an n-dimensional simple comodule over H_{2n^2}, (W_n, δ), with $W_n = \Bbbk\{w_0, \ldots, w_{n-1}\}$ and coaction $\delta : W_n \to H_{2n^2} \otimes W_n$ given by:*

$$\delta(w_i) = \frac{1}{n^2} \sum_{r,s} q^{si} |r, s\rangle_z \otimes w_{i+r} = \frac{1}{n} \sum_{a,b} q^{-ab} x^a y^i z \otimes w_{i+b}.$$

Proof. We have already seen that W_n is n-dimensional and simple. The identity is straightforward to check. □

4.4. **The category of comodules over $\mathbf{H_{2n^2}}$.** H_{2n^2} is a finite-dimensional semisimple Hopf algebra [15]. As such, $H_{2n^2}^*$ is also a semisimple finite-dimensional algebra. Thus, the Artin-Wedderburn theorem tells us for $\{V_i\}$ an indexing of the simple modules over $H_{2n^2}^*$, we have:

$$\sum_i \dim(V_i)^2 = \dim(H_{2n^2}) = 2n^2.$$

Applying the equivalence of comodules over H_{2n^2} and modules over $H_{2n^2}^*$, this tells us for $\{C_i\}$ an indexing of the simple comodules over H_{2n^2}, we have:

$$\sum_i \dim(C_i)^2 = 2n^2.$$

We have found n^2 1-dimensional, non-isomorphic, simple comodules alongside a single n-dimensional simple comodule. Thus, we have:

$$\underbrace{1^2 + 1^2 + \cdots + 1^2}_{n^2 \text{ terms}} + n^2 = n^2(1) + n^2 = 2n^2.$$

Therefore, we have found all simple comodules over H_{2n^2}.

We now compute the category of simple comodules over H_{2n^2}.

Theorem 4.7. *The category ${}^{H_{2n^2}}\mathcal{M}$ of comodules over H_{2n^2} is a Tambara-Yamagami category over $\mathbb{Z}_n \otimes \mathbb{Z}_n$. The simple comodules are as follows:*

1. *There are n^2 simple one-dimensional comodules $W_{(a,b)} = \Bbbk\{w_{(a,b)}\}$ with coaction:*
$$\delta(w_{(a,b)}) = x^a y^b \otimes w_{(a,b)},$$
for $(a,b) \in \mathbb{Z}_n \times \mathbb{Z}_n$.
2. *There is one simple n-dimensional comodule $W_n = \Bbbk\{w_0, \dots, w_{n-1}\}$ with coaction:*
$$\delta(w_i) = \frac{1}{n}\sum_{a,b} q^{-ab} x^a y^i z \otimes w_{i+b}.$$

Proof. We have already shown that the set $\{W_{(a,b)}\}_{(a,b)\in\mathbb{Z}_n} \coprod \{W_n\}$ is exactly the simple comodules over H_{2n^2}. From Definition 2.14, it suffices to check the Tambara-Yamagami fusion rules to complete the proof.

Given two one-dimensional simple objects $W_{(a,b)}, W_{(c,d)}$, consider the product $W_{(a,b)} \otimes W_{(c,d)}$. We have:

$$\delta(w_{(a,b)} \otimes w_{(c,d)}) = \left(x^a y^b \cdot x^c y^d\right) \otimes (w_{(a,b)} \otimes w_{(c,d)}) = x^{a+c} y^{b+d} \otimes (w_{(a,b)} \otimes w_{(c,d)}).$$

Thus, as comodules, $W_{(a,b)} \otimes W_{(c,d)} \cong W_{(a+c,b+d)}$ by the map $w_{(a,b)} \otimes w_{(c,d)} \to w_{(a+c,b+d)}$.

Given a one-dimensional object $W_{(a,b)}$ and an n-dimesnional object W_n, consider the product $W_{(a,b)} \otimes W_n$. For illustrative purposes, we compute the isomorphism in detail.

We have:

$$\begin{aligned}\delta(w_{(a,b)}\otimes w_i) &= \frac{1}{n}\sum_{c,d} q^{-cd}x^ay^bx^cy^iz\otimes w_{i+d}\\ &= \frac{1}{n}\sum_{c,d} q^{-cd}x^{a+c}y^{b+i}z\otimes w_{i+d}\\ &= \frac{1}{n}\sum_{c',d'} q^{-(c'-a)(d'+b)}x^{c'}y^{i+b}z\otimes w_{i+b+d'}.\end{aligned}$$

So, define $\varphi : W_{(a,b)}\otimes W_n \to W_n$ by $\varphi(w_{(a,b)}\otimes w_i) = q^{-ai}w_{i+b}$. Clearly, this is a bijection of bases, so we must check it is a comodule morphism, e.g $(\mathrm{id}\otimes\varphi)\circ\delta = \delta\circ\varphi$. We get:

$$\begin{aligned}(\mathrm{id}\otimes\varphi)(\delta(w_{(a,b)}\otimes w_i)) &= \frac{1}{n}\sum_{c,d} q^{-cd}x^{a+c}y^{b+i}z\otimes\varphi(w_{i+d})\\ &= \frac{1}{n}\sum_{c,d} q^{-cd}q^{-a(i+d)}x^{a+c}y^{b+i}z\otimes w_{i+b+d}\\ &= \frac{1}{n}\sum_{c',d} q^{-c'd-ai}x^{c'}y^{b+i}z\otimes w_{i+b+d},\end{aligned}$$

where in the third equality we substitute $c' = c+a$. We also have:

$$\delta(\varphi(w_{(a,b)}\otimes w_i)) = \delta(q^{-ai}w_{i+b}) = \frac{1}{n}\sum_{c,d} q^{-ai}q^{-cd}x^cy^{i+b}z\otimes w_{i+b+d}.$$

Identifying c' and c shows that $W_{(a,b)}\otimes W_n \cong W_n$. Similarly, we see $W_n\otimes W_{(a,b)}\cong W_n$.

Finally, consider the product $W_n\otimes W_n$ of two n-dimensional objects. One may check:

$$\delta(w_i\otimes w_j) = \frac{q^{-ij}}{n^2}\sum_{u,v}\sum_{r,s} q^{-ur+ui-vs+vj}x^uy^v\otimes(q^{rs}w_r\otimes w_s)$$

Now define $\Psi : W_n\otimes W_n\to\bigoplus_{a,b\in\mathbb{Z}_n}W_{(a,b)}$ and $\Phi : \bigoplus_{a,b\in\mathbb{Z}_n}W_{(a,b)}\to W_n$ by:

$$\Psi(w_r\otimes w_s) = \frac{1}{n}\sum_{a,b} q^{-rs+ar+bs}w_{(a,b)},$$

$$\Phi(w_{(a,b)}) = \frac{1}{n}\sum_{i,j} q^{-ai-bj+ij}w_i\otimes w_j.$$

It is straightforward to check Ψ and Φ are inverse comodule maps. So, we conclude

$$W_n\otimes W_n\cong\bigoplus_{a,b\in\mathbb{Z}_n}W_{(a,b)}.$$

□

Remark 4.8. Recall that the Pansera algebras may also be described as a $\mathbb{Z}_2$ extension of the group algebra $\mathbb{Z}_n \times \mathbb{Z}_n$ [15]. Using the theory of abelian extensions, the methods in [21] and [9] can be used to abstractly show that ${}^{H_{2n^2}}\mathcal{M}$ is a Tambara-Yamagami category. However, this approach does not give explicit descriptions of the comodules.

4.5. **Induced coaction.** Given a Hopf algebra H and a simple (left) H-module L, recall from Lemma 2.24 that the induced coaction on $L\otimes H$ is given by $\delta(l\otimes h) = h_{(1)}\otimes(l\otimes h_{(2)})$. The following is straightforward to check in the manner of previous computations:

Lemma 4.9. *Given a simple (left) H_{2n^2}-module L, the induced coaction δ on $L \otimes H_{2n^2}$ is given by:*

$$\delta(l \otimes x^a y^b) = x^a y^b \otimes (l \otimes x^a y^b),$$
$$\delta(l \otimes x^a y^b z) = \frac{1}{n^2} \sum_{e,f} q^{ea+fb} \left|e, f\right\rangle_z \otimes (l \otimes x^a y^{b+e} z),$$

for all $l \in L$.

5. Yetter–Drinfeld Modules over H_{2n^2}

5.1. **The category of Yetter–Drinfeld modules over $\mathbf{H_{2n^2}}$.** Recall the following classical result, which may be found as Theorem 4.1.6 in [10]:

Proposition 5.1. *On the level of monoidal categories, the category of Yetter–Drinfeld modules of a Hopf algebra H is isomorphic to the center of the category of comodules over H. That is,*

$$ {}^H_H\mathcal{YD} \cong \mathcal{Z}\left({}^H\mathcal{M}\right).$$

In particular, since the category of comodules of H_{2n^2} is Tambara–Yamagami, the following, which appears in more detail as Proposition 4.1 in [9], holds:

Proposition 5.2 (Gelaki et al.)**.** *[(i)] Given $\mathcal{TY}(G, \chi, \tau)$ a Tambara–Yamagami category, the following is a list of the number of simple objects in $\mathcal{Z}(\mathcal{TY}(G, \chi, \tau))$ with a given dimension up to isomorphism:*

(1) $2m$ one-dimensional objects,
(2) $\frac{m(m-1)}{2}$ two-dimensional objects,
(3) $2m$ $\sqrt{m}$-dimensional objects,

where $m = |G|$ and where the dimension is the Frobenius-Perron dimension.

Combining these, we see ${}^{H_{2n^2}}_{H_{2n^2}}\mathcal{YD} \cong \mathcal{Z}({}^{H_{2n^2}}\mathcal{M})$ may be realized as the center of the Tambara–Yamagami category of comodules, with the group $G = \mathbb{Z}_n \times \mathbb{Z}_n$. Furthermore, at the level of a monoidal category, we know by Corollary 2.23 that ${}^{H_{2n^2}}_{H_{2n^2}}\mathcal{YD} \cong \mathrm{Rep}(D(H_{2n^2}))$ where $D(H_{2n^2})$ is the Drinfeld double of H_{2n^2}. Thus, all objects in ${}^{H_{2n^2}}_{H_{2n^2}}\mathcal{YD}$

may be realized as modules over $D(H)$, and thus, by Remark 2.20, their Frobenius-Perron dimension coincides with their $\Bbbk$-vector space dimension.

We ultimately arrive at:

Lemma 5.3. *The following is a list of the number of non-isomorphic simple Yetter–Drinfeld modules in* ${}^{H_{2n^2}}_{H_{2n^2}}\mathcal{YD}$ *for a given dimension:*

(1) There are $2n^2$ one-dimensional non-isomorphic simple Yetter–Drinfeld modules.
(2) There are $\frac{n^2(n^2-1)}{2}$ two-dimensional non-isomorphic simple Yetter–Drinfeld modules.
(3) There are $2n^2$ n-dimensional non-isomorphic simple Yetter–Drinfeld modules.

Lemma 2.24 implies that simple Yetter-Drinfeld over H_{2n^2} modules arise as the simple Yetter-Drinfeld submodules of $L \otimes H_{2n^2}$ for L a (left) H-module. In Section 3, we computed the module actions on such $L \otimes H$. In Section 4, we determined all comodules. We will now combine these analyses to prove that certain submodules of these $L \otimes H_{2n^2}$ are simple, thus determining some simple Yetter-Drinfeld modules over H_{2n^2}.

5.2. **One-dimensional simple Yetter–Drinfeld Modules.** Take the induced module $T_k^{\pm} \otimes H_{2n^2}$. Consider the subspace $U_a^{k,\pm} = \Bbbk\{u_a^{k,\pm}\}$ with $u_a^{k,\pm} = t_k^{\pm} \otimes x^a y^{a-k}$. From Lemma 3.1, Lemma 3.2, and Lemma 4.9, a direct computation shows $U_a^{k,\pm}$ has an H_{2n^2}-action by:

$$x \cdot u_a^{k,\pm} = q^k u_a^{k,\pm}, \quad y \cdot u_a^{k,\pm} = q^k u_a^{k,\pm}, \quad z \cdot u_a^{k,\pm} = \pm p^{k^2} u_a^{k,\pm},$$

and coaction δ given by:

$$\delta(u_a^{k,\pm}) = x^a y^{a-k} \otimes u_a^{k,\pm}.$$

Since $T_k^{\pm} \otimes H_{2n^2}$ is an induced Yetter–Drinfeld module and $U_a^{k,\pm}$ is closed under the action and coaction, we know $U_a^{k,\pm}$ is a one-dimensional Yetter–Drinfeld submodule. Each $U_a^{k,\pm}$ is non-isomorphic since if the sign $\pm$ or the parameter k differs, they are non-isomorphic as modules, and if the parameter a differs, they are non-isomorphic as comodules.

As $a, k \in \mathbb{Z}_n$, $\pm \in \{1, -1\}$, this gives $2n^2$ non-isomorphic, 1-dimensional Yetter–Drinfeld modules, and by Proposition 5.2, this must be all simple 1-dimensional Yetter–Drinfeld modules, giving:

Proposition 5.4. *The one-dimensional simple Yetter–Drinfeld modules over H_{2n^2} are given by $U_a^{k,\pm} = \Bbbk\{u_a^{k,\pm}\}$ for $a, k \in \mathbb{Z}_n$, $\pm \in \{1, -1\}$, with the H_{2n^2}-action given by:*

$$x \cdot u_a^{k,\pm} = q^k u_a^{k,\pm}, \quad y \cdot u_a^{k,\pm} = q^k u_a^{k,\pm} \quad z \cdot u_a^{k,\pm} = \pm p^{k^2} u_a^{k,\pm}$$

and coaction δ given by:

$$\delta(u_a^{k,\pm}) = x^a y^{a-k} \otimes u_a^{k,\pm}.$$

All such modules are distinct.

5.3. **Two-dimensional simple Yetter–Drinfeld modules.** Consider the induced module $T_k^+ \otimes H_{2n^2}$. For $a, b \in \mathbb{Z}_n$ such that $a \neq b+k \pmod n$, let $v_{a,b}^{k,1} = t_k^+ \otimes x^a y^b$ and $v_{a,b,c,d}^{k,2} = t_k^+ \otimes x^{b+k} y^{a-k}$.

Consider the subspace $V_{a,b}^k = \Bbbk\{v_{a,b}^{k,1}, v_{a,b}^{k,2}\}$. This is two-dimensional as $t_k^+ \otimes x^a y^b, t_k^+ \otimes x^{b+k} y^{a-k}$ are distinct basis elements of $T_k^+ \otimes H_{2n^2}$ for $a \neq b+k \pmod n$. The H_{2n^2}-actions are given by:

$$x \cdot v_{a,b}^{k,1} = q^k v_{a,b}^{k,1} \qquad x \cdot v_{a,b}^{k,2} = q^k v_{a,b}^{k,2}$$

$$y \cdot v_{a,b}^{k,1} = q^k v_{a,b}^{k,1} \qquad y \cdot v_{a,b}^{k,2} = q^k v_{a,b}^{k,2}$$

$$z \cdot v_{a,b}^{k,1} = p^{k^2} v_{a,b}^{k,2} \qquad z \cdot v_{a,b}^{k,2} = p^{k^2} v_{a,b}^{k,1}[4pt]$$

and the coaction δ is given by:

$$\delta(v_{a,b}^{k,1}) = x^a y^b \otimes v_{a,b}^{k,1} \qquad \delta(v_{a,b}^{k,2}) = x^c y^d \otimes v_{a,b}^{k,2}$$

Since $T_k^+ \otimes H_{2n^2}$ is an induced Yetter–Drinfeld module and $V_{a,b}^k$ is closed under the action and coaction, we conclude $V_{a,b}^k$ is indeed a two-dimensional Yetter–Drinfeld submodule.

To see that $V_{a,b}^k$ is simple, let $u_{a,b}^{k,1} = v_{a,b}^{k,1} + v_{a,b}^{k,2}$ and $u_{a,b}^{k,2} = v_{a,b}^{k,1} - v_{a,b}^{k,2}$. Then:

$$x \cdot u_{a,b}^{k,1} = q^k u_{a,b}^{k,1}, \qquad x \cdot u_{a,b}^{k,2} = q^k u_{a,b}^{k,2},$$

$$y \cdot u_{a,b}^{k,1} = q^k u_{a,b}^{k,1}, \qquad y \cdot u_{a,b}^{k,2} = q^k u_{a,b}^{k,2},$$

$$z \cdot u_{a,b}^{k,1} = p^{k^2} u_{a,b}^{k,1}, \qquad z \cdot u_{a,b}^{k,2} = -p^{k^2} u_{a,b}^{k,2},$$

and:

$$\delta(u_{a,b}^{k,1}) = x^a y^b \otimes (u_{a,b}^{k,1} + u_{a,b}^{k,2}) + x^c y^d \otimes (-u_{a,b}^{k,1} + u_{a,b}^{k,2}),$$

$$\delta(u_{a,b}^{k,2}) = x^a y^b \otimes (u_{a,b}^{k,1} + u_{a,b}^{k,2}) + x^c y^d \otimes (u_{a,b}^{k,1} - u_{a,b}^{k,2}).$$

Thus, the module action is diagonal with distinct eigenvalues. So, any one-dimensional submodule must be an eigenspace. But on each eigenspace, the coaction is not diagonal. So, we conclude $V_{a,b}^k$ is simple with module structure $T_k^+ \otimes T_k^-$ and comodule structure $W_{(a,b)} \oplus W_{b+k,a-k}$.

We can see $V_{a,b}^{k_1} \cong V_{c,d}^{k_2}$ if and only if $k_1 = k_2$ and either $(a,b) = (c,d)$ or $(a,b) = (d+k, c-k)$ in the following way. Comparing the module strucutres $T_{k_1}^+ \otimes T_{k_1}^-$ and $T_{k_2}^+ \otimes T_{k_2}^-$, we realize $T_{k_1}^+ \otimes T_{k_1}^- \cong T_{k_2}^+ \otimes T_{k_2}^-$ if and only if $k_1 = k_2$. Comparing the comodule structures, if $k_1 = k_2 = k$, we conclude $W_{(a,b)} \otimes W_{(b+k,a-k)}$ and $W_{(c,d)} \otimes W_{(d+k,c-d)}$ are isomorphic if and only if $(a,b) = (c,d)$ or $(a,b) = (d+k, c-d)$. It is straightforward to check that when $k_1 = k_2 = k$ and $(a,b) = (c,d)$ or $(a,b) = (d+k, c-k)$ that $V_{a,b}^{k_1} \cong V_{c,d}^{k_2}$. This finishes the argument.

How many non-isomorphic modules of the form $V_{a,b}^k$ are there? There are n choices for k, n choices for a, and $(n-1)$ choices for b such that $b \neq a+k \pmod n$. But we must divide by two to account for the fact that $V^k_{a,b} \cong V^k_{b+k,a-k}$. So we end up with $\frac{n^2(n-1)}{2}$ distinct objects.

In the same manner, we may construct simple two-dimensional modules Yetter-Drinfeld arising from the $\pi(i,j) \otimes H_{2n^2}$ by taking $V^{i,j}_{a,b} = \Bbbk\{v_1 \otimes x^a y^b, v_2 \otimes x^{b+i} y^{a-j}\}$. We may similarly check that the modules $V^{i,j}_{a,b}$ and $V^{i',j'}_{c,d}$ are isomorphic if and only if $i = i'$, $j = j'$.

How many objects of the form $V^{i,j}_{a,b}$ are there? There are $\frac{n(n-1)}{2}$ choices for the pair (i,j), since we must have $i < j$. There are n choices for a and n choices for b. So we have $\frac{n^3(n-1)}{2}$ distinct objects.

But now, we have found $\frac{n^4-n^3}{2} + \frac{n^3-n^2}{2} = \frac{n^2(n^2-1)}{2}$ non-isomorphic two-dimensional simple Yetter–Drinfeld modules, and by Proposition 5.2, this is all such objects. Thus:

Proposition 5.5. *The simple two-dimensional Yetter–Drinfeld modules over H_{2n^2} are as follows.*

(1) *For $k \in \mathbb{Z}_n$ and $a, b \in \mathbb{Z}_n$ such that $a \neq b + k$, there are simple two-dimensional modules $V^k_{a,b} = \Bbbk\{\, v^{k,1}_{a,b},\, v^{k,2}_{a,b} \,\}$ is a module with action and coaction:*

$$\begin{aligned} x \cdot v^{k,1}_{a,b} &= q^k v^{k,1}_{a,b}, & x \cdot v^{k,2}_{a,b} &= q^k v^{k,2}_{a,b}, \\ y \cdot v^{k,1}_{a,b} &= q^k v^{k,1}_{a,b}, & y \cdot v^{k,2}_{a,b} &= q^k v^{k,2}_{a,b}, \\ z \cdot v^{k,1}_{a,b} &= p^{k^2} v^{k,2}_{a,b}, & z \cdot v^{k,2}_{a,b} &= p^{k^2} v^{k,1}_{a,b}, \\ \delta(v^{k,1}_{a,b}) &= x^a y^b \otimes v^{k,1}_{a,b}, & \delta(v^{k,2}_{a,b}) &= x^{b+k} y^{a-k} \otimes v^{k,2}_{a,b}. \end{aligned}$$

The modules $V^k_{a,b}$ and $V^{k'}_{c,d}$ are isomorphic if and only if $k = k'$ and $(a,b) = (c,d)$ or $(a,b) = (d+k, c-k)$.

(2) *For $a, b, i, j \in \mathbb{Z}_n$ with $i < j$, $V^{i,j}_{a,b} = \Bbbk\{\, v^{i,j,1}_{a,b},\, v^{i,j,2}_{a,b} \,\}$ is a module with action and coaction:*

$$\begin{aligned} x \cdot v^{i,j,1}_{a,b} &= q^i v^{i,j,1}_{a,b}, & x \cdot v^{i,j,2}_{a,b} &= q^j v^{i,j,2}_{a,b}, \\ y \cdot v^{i,j,1}_{a,b} &= q^j v^{i,j,1}_{a,b}, & y \cdot v^{i,j,2}_{a,b} &= q^i v^{i,j,2}_{a,b}, \\ z \cdot v^{i,j,1}_{a,b} &= q^{ij} v^{i,j,2}_{a,b}, & z \cdot v^{i,j,2}_{a,b} &= v^{i,j,1}_{a,b}, \\ \delta(v^{i,j,1}_{a,b}) &= x^a y^b \otimes v^{i,j,1}_{a,b}, & \delta(v^{i,j,2}_{a,b}) &= x^{b+i} y^{a-j} \otimes v^{i,j,2}_{a,b}. \end{aligned}$$

5.4. **n-dimensional simple Yetter–Drinfeld modules.** The simple n-dimensional Yetter–Drinfeld modules are more complicated. We provide only a partial classification.

Consider the module $T_0^{\pm}$ and let:

$$v_b^{l,\pm} = \sum_a p^{a^2} q^{a(b+l)} (t_0^{\pm} \otimes x^a y^b z).$$

Consider the subspace $V_n^{l,\pm} = \{v_0^{l,\pm}, \dots, v_{n-1}^{l,\pm}\}$. We wish to show this is a Yetter–Drinfeld module with comodule structure W_n. Checking the coaction, we see:

$$\begin{aligned}
\delta(v_b^{l,\pm}) &= \sum_a p^{a^2} q^{a(b+l)}\, \delta(t_0^\pm \otimes x^a y^b z) \\
&= \frac{1}{n^2} \sum_{a,e,f} q^{ea+fb} p^{a^2} q^{a(b+l)}\ |e,f\rangle_z \otimes \left(t_0^\pm \otimes x^a y^{b+e} z\right) \\
&= \frac{1}{n^2} \sum_{e,f} q^{fb}\, |e,f\rangle_z \otimes \left(\sum_a (q^{a(b+e+l)} p^{a^2}) \left(t_0^\pm \otimes x^a y^{b+e} z\right)\right) \\
&= \frac{1}{n^2} \sum_{e,f} q^{fb}\, |e,f\rangle_z \otimes v_{b+e}^{l,\pm}.
\end{aligned}$$

Now, we check the module actions. We have:

$$\begin{aligned}
x \cdot v_b^{l,\pm} &= \sum_a p^{a^2} q^{a(b+l)} \left(x \cdot (t_0^\pm \otimes x^a y^b z)\right) \\
&= \sum_a p^{a^2} q^{a(b+l)} \left(t_0^\pm \otimes x^{a+1} y^{b-1} z\right) \\
&= \sum_{a'} p^{a'^2} p^{-2a'} q^{a'(b-1+l)} q^{-(b-1+l)} q^{a'} q^{-1} \left(t_0^\pm \otimes x^{a'} y^{b-1} z\right) \\
&= p\, q^{-(b+l)} v_{b-1}^{l,\pm}.
\end{aligned}$$

where $a' = a+1$. Similarly, $y \cdot v_b^{l,\pm} = p\, q^{b+l} v_{b+1}^{l,\pm}$.

The z-action is:

$$\begin{aligned}
z \cdot v_b^{l,\pm} &= \sum_a p^{a^2} q^{a(b+l)} \left(z \cdot (t_0^\pm \otimes x^a y^b z)\right) \\
&= \frac{\pm 1}{n} \sum_{c,d,a} p^{a^2} q^{a(b+l)} q^{-cd-d^2}\, x^{b+c} y^{a+d} z \\
&= \frac{\pm 1}{n} \sum_{c',d',a} q^{-(c'-b)(d'-a)-(d'-a)^2} p^{a^2} q^{a(b+l)} x^{c'} y^{d'} z,
\end{aligned}$$

where we substitute $c' = b+c$, $d' = a+d$ in the third line. Now, using $q = p^2$, we get:

$$z \cdot v_b^{l,\pm} = \frac{\pm 1}{n} \sum_{d'} p^{2bd'-2(d')^2} \sum_{a,c'} p^{2c'a-a^2} p^{4ad'+2al} \left(p^{-2c'd'} x^{c'} y^{d'} z\right).$$

Notice that $p^{2c'a-a^2} = p^{-(a-c')^2+(c')^2}$. Plugging this in and substituting $a' = a - c'$, we obtain:

$$\begin{aligned} z \cdot v_b^{l,\pm} &= \frac{\pm 1}{n} \sum_{d'} p^{2bd'-2(d')^2} \sum_{a',c'} p^{-(a')^2+(c')^2} p^{4(a'+c')d'+2(a'+c')l} \left(p^{-2c'd'} x^{c'} y^{d'} z\right) \\ &= \frac{\pm 1}{n} \sum_{d'} p^{2bd'-2(d')^2} \sum_{a'} p^{-(a')^2+4a'd'+2a'l} \sum_{c'} p^{(c')^2+2c'd'+2c'l} x^{c'} y^{d'} z \\ &= \frac{\pm 1}{n} \sum_{d'} q^{bd'} q^{-(d')^2} \sum_{a'} p^{-(a')^2} q^{2a'd'} q^{a'l} v_{d'}^{l,\pm}. \end{aligned}$$

We have thus seen that $V_n^{l,\pm}$ is closed under the action and coaction, and is thus a sub-Yetter–Drinfeld module of $T_0^+ \otimes H_{2n^2}$. As a comodule, $V_n^{l,\pm}$ is isomorphic to the n-dimensional simple comodule W_n. Thus, as a comodule, any nontrivial sub-Yetter–Drinfeld module V' of $V_n^{l,\pm}$ must be a nontrivial subcomodule of W_n and thus equal to W_n. Then V' must be n-dimensional, and since $V_n^{l,\pm}$ is n-dimensional, we see $V' \cong V_n^{l,\pm}$. Thus, $V_n^{l,\pm}$ must be simple as a Yetter–Drinfeld module.

Combining the results of this analysis, we recover:

Proposition 5.6. *For $\pm \in \{-1, 1\}$ and $l \in \mathbb{Z}_n$, there are simple n-dimensional Yetter–Drinfeld modules over H_{2n^2} by $V_n^{l,\pm} = \Bbbk\{v_0^{l,\pm}, \dots, v_{n-1}^{l,\pm}\}$, with H_{2n^2}-action by:*

$$\begin{aligned} x \cdot v_i^{l,\pm} &= pq^{-(i+l)} v_{i-1}^{l,\pm} \\ y \cdot v_i^{l,\pm} &= pq^{i+l} v_{i+1}^{l,\pm} \\ z \cdot v_i^{l,\pm} &= \frac{\pm 1}{n} \sum_{a,b} q^{ia} q^{-(a)^2} p^{-(b)^2} q^{2ab} q^{bl} v_a^{l,\pm}, \end{aligned}$$

and coaction by:

$$\delta(v_i^{l,\pm}) = \frac{1}{n^2} \sum_{r,s} q^{si} \left|r, s\right\rangle_z \otimes v_{i+r}^{l,\pm}.$$

We now diagonalize the x-action by $w_j^{l,\pm} = \sum_i p^{i^2} q^{i(j+l)} v_i^{l,\pm}$. We have:

$$\begin{aligned} x \cdot w_j^{l,\pm} &= \sum_i p^{i^2} q^{i(j+l)} \left(x \cdot v_i^{l,\pm}\right) \\ &= \sum_i p^{i^2} q^{i(j+l)} pq^{-i-l} v_{i-1}^{l,\pm} \\ &= \sum_i p^{i^2-2i+1} q^{(i-1)(j+l)} \cdot q^j v_{i-1}^{l,\pm} = q^j w_j^{l,\pm}. \end{aligned}$$

Similarly $y \cdot w_j^{l,\pm} = q^{-j} w_j^{l,\pm}$. So, since we have found n eigenvectors with distinct eigenvalues, we conclude the map $v \to x \cdot v$ has no repeated eigenvalues.

We then see:

$$\begin{aligned}
z \cdot w_j^{l,\pm} &= \sum_i p^{i^2} q^{(j+l)i} (z \cdot v_i^{l,\pm}) \\
&= \sum_i p^{i^2} q^{(j+l)i} \left(\frac{\pm 1}{n} \sum_a q^{ai} q^{-a^2} \left(\sum_b p^{-b^2} q^{2ab} q^{bl} \right) v_a^{l,\pm} \right) \\
&= \frac{\pm 1}{n} \sum_a \left(q^{-a^2} \left(\sum_i p^{i^2} q^{(j+l)i} q^{ai} \right) \left(\sum_b p^{-b^2} q^{2ab} q^{bl} \right) \right) v_a^{l,\pm}.
\end{aligned}$$

For general n, it is not possible to simplify in a useful way from here. Restricting to the case of n even, notice:

$$p^{(i+n)^2} = p^{i^2} p^{2in} p^{n^2} = p^{i^2} \left(p^{2n}\right)^i (p^n)^n = p^{i^2} \cdot 1 \cdot (-1)^n,$$

and thus for even n, $p^{(i+n)^2}$ is periodic modulo n. This implies that the sum $G_k = \sum_{i=0}^{n-1} p^{(i+k)^2}$ always has the same value, since the consecutive residues $k, \ldots, k+n-1$ may be shifted to sum over the residues $0, \ldots, n-1$. Denote this common value G. Now notice:

$$\sum_i p^{i^2} q^{(j+l+a)i} = p^{-(j+l+a)^2} \left(\sum_i p^{(i-j-l-a)^2} \right) = p^{-(j+l+a)^2} G.$$

Because of this, $G'_k = \sum_{j=0}^{n-1} p^{-(j+k)^2}$ is invariant with common value $G' = \sum_{b=0}^{n-1} p^{-b^2}$. We get:

$$\sum_b p^{-b^2} q^{2ab} q^{bl} = p^{(2a+l)^2} \sum_b p^{-(b-2a-l)^2} = p^{(2a+l)^2} G',$$

where G' is the common value of the G'_k. Plugging in, one can check:

$$\begin{aligned}
z \cdot w_j^{l,\pm} &= \frac{\pm 1}{n} \sum_a q^{-a^2} p^{-(j+l+a)^2} p^{(2a+l)^2} G G' v_a^{l,\pm} \\
&= \frac{\pm 1}{n} \sum_a p^{a^2} q^{a(l-j)} \left(p^{-j^2} q^{-jl} G G' \right) v_a^{l,\pm} \\
&= \frac{\pm 1}{n} p^{-j^2} q^{-jl} G G' w_{-j}^{l,\pm}.
\end{aligned}$$

Thus, we may now check that the $V_n^{l,\pm}$ are non-isomorphic. Suppose there is an isomorphism $\Psi : V_n^{l,s} \to V_n^{l',s'}$ where $s, s' \in \{1, -1\}$. Then, recall that the eigenvalues of the operator $X : v \to x \cdot v$ are all distinct, with $w_j^{l,s}$ the unique (up to scalar multiple) eigenvector of eigenvalue q^j. Any isomorphism Ψ must send an eigenvector of X to an eigenvector of X of the same eigenvalue since isomorphisms of modules commute with the x and y module actions. Thus, for $w_j^{l,s}$ in $V_n^{l,s}$ and $w_j^{l',s'}$ in $V_n^{l,s'}$, we see $\Psi(w_j^{l,s}) = c_j w_j^{l',s'}$ for some constant c_j. But then, since the z-action also commutes with Ψ, we have:

$$\begin{aligned}\Psi(z \cdot w_j^{l,s}) &= \Psi\left(\frac{s}{n}p^{-j^2-jl}GG'w_{-j}^{l,s}\right)\\ &= c_j\frac{s}{n}p^{-j^2}q^{-jl}GG'w_{-j}^{l',s'},\end{aligned}$$

and:

$$\begin{aligned}z \cdot \Psi(w_j^{l,s}) &= c_j\left(z \cdot w_j^{l',s'}\right)\\ &= c_j\frac{s'}{n}p^{-j^2}q^{-jl'}GG'w_{-j}^{l',s'}.\end{aligned}$$

Setting these equal, we see:

$$c_j\frac{s}{n}p^{-j^2}q^{-jl}GG' = c_j\frac{s'}{n}p^{-j^2}q^{-jl'}GG',$$

which simplifies to $(s/s')q^{-j(l-l')} = 1$. This must hold for all j. Taking $j = 0$, we must have $s/s' = 1$ or $s = s'$. This also forces $q^{-j(l-l')} = 1$. Then, summing over all j, we see:

$$\sum_j q^{-j(l-l')} = n\delta_{0,l-l'},$$

by Lemma 2.26. But, since each individual term $q^{-j(l-l')}$ is 1, the sum is n. So $\delta_{0,l-l'} = 1$ and we see $l = l'$. So, if $V_n^{l,s} \cong V_n^{l,s'}$, $l = l'$ and $s = s'$. Thus, we have:

Proposition 5.7. *The n-dimensional simple Yetter–Drinfeld modules $V_n^{l,\pm}$ are non-isomorphic for all $l \in \mathbb{Z}_n$ when n is even.*

Testing numerically, we pose the following conjecture:

Conjecture 5.8. *The n-dimensional simple Yetter–Drinfeld modules $V_n^{l,\pm}$ are non-isomorphic for all $l \in \mathbb{Z}_n$ when n is odd.*

We constructed the $V_n^{l,\pm}$ from the z-part of $T_0^{\pm} \otimes H_{2n^2}$. We conjecture more generally:

Conjecture 5.9. *The subspace of $T_k^{\pm} \otimes H_{2n^2}$ with basis $\{t_k^{\pm} \otimes x^a y^b z\}$ furnishes n distinct simple n-dimensional Yetter–Drinfeld modules isomorphic to the $V_n^{l,\pm}$.*

6. Conclusions and Future Work

As discussed in Section 1, the main work of future interest is to analyze the Nichols algebras over H_{2n^2}, with the ultimate goal of applying the Andruskiewitsch-Schneider lifting method to compute all finite-dimensional Hopf algebras over H_{2n^2}. Another interesting direction would be to try to extend this analysis to Lomp's algebras $H_{n,m}$.

To analyze the Nichols algebras, including the potential Hopf algebra extending $U_q(\mathfrak{sl}_3)$ at q an nth primitive root of unity, we must complete the current research program by determining all simple Yetter–Drinfeld modules over H_{2n^2}. We also hope to compute the modular data of the category of comodules. This will abstractly provide the modular data for the category of Yetter–Drinfeld modules. Concretely computing the fusion rules for the Yetter–Drinfeld modules would also be interesting.

Regarding the results presented in this paper, the fact that the category of comodules is Tambara–Yamagami is exciting. Tambara–Yamagami categories are a rare class of fusion categories where the associativity data can be described explicitly, and thus serve as good toy examples to test larger ideas in tensor categories. Since the Pansera algebra H_{2n^2} carries historical importance and is tractable for study, our results lend credence to the idea that the Pansera algebra H_{2n^2} is a good toy example to test more general ideas in Hopf algebras.

In this paper, we proposed Conjecture 5.8. We also pose the following conjecture:

Conjecture 6.1. *Each induced Yetter–Drinfeld module $\pi(i,j) \otimes H$ furnishes four non-isomorphic n-dimensional simple Yetter–Drinfeld modules over H_{2n^2}.*

This will provide $4 \cdot \frac{n(n-1)}{2} = 2n(n-1)$ n-dimensional Yetter–Drinfeld modules. Combined with the $2n$ n-dimensional simple Yetter–Drinfeld modules from Conjecture 5.8, we will have $2n^2 - 2n + 2n = 2n^2$ n-dimensional simple Yetter–Drinfeld modules. By Proposition 5.2, this would be all the Yetter–Drinfeld modules over H_{2n^2}, thus completing the classification.

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