

Quantum Groups from a Fomin-Kirillov Algebra

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Quantum groups were introduced by Drinfeld and Jimbo in the 1980s as algebraic objects providing deformations of classical Lie algebras and function algebras on groups. They have since found applications in knot theory, quantum field theory, and noncommutative geometry. Quantum groups are useful because:

- (1) They generalize symmetry in the noncommutative setting.
- (2) They link algebra, geometry, and physics through deformation theory.

My work focuses on constructing a new quantum function algebra through **codouble bosonization**.

The most common definition of a quantum group is as a Hopf algebra that deforms the structure of a classical group or Lie algebra. This includes two main species:

- (1) Quantum enveloping algebras $U_q(\mathfrak{g})$ deform $U(\mathfrak{g})$.
- (2) Quantum function algebras $\mathcal{O}_q(G)$ deform $\mathcal{O}(G)$.

Quantum groups can be decomposed into three parts, this comes from way we typically decompose Lie algebras:

$$U_q(\mathfrak{g}) = U_q(\mathfrak{n}^-) \otimes U_q(\mathfrak{h}) \otimes U_q(\mathfrak{n}^+)$$

Importantly, $U_q(\mathfrak{n}^+)$ is a Nichols algebra built from a root vector space V and $U_q(\mathfrak{n}^-)$ is dual to $U_q(\mathfrak{n}^+)$.

A **braided Hopf algebra** lives in a braided monoidal category, such as the Yetter–Drinfeld category ${}^H_H\mathcal{YD}$ of modules and comodules over a Hopf algebra H . For all of the compatibility diagrams (example below) that a normal hopf algebra needs to satisfy, the braiding $\sigma(v \otimes w)$ replaces the standard swap map $\tau(v \otimes w) = w \otimes v$.

$$\begin{array}{ccccc}
 B \otimes B & \xrightarrow{\mu} & B & \xrightarrow{\Delta} & B \otimes B \\
 \Delta \otimes \Delta \downarrow & & & & \uparrow \mu \otimes \mu \\
 B \otimes B \otimes B \otimes B & \xrightarrow{\text{id} \otimes \tau \otimes \text{id}} & B \otimes B \otimes B \otimes B & &
 \end{array}$$

We will now introduce an algebra that can be thought of as a special braided Hopf algebra in the Yetter–Drinfeld category ${}^{\mathbb{k}S_3}_{\mathbb{k}S_3}\mathcal{YD}$, which will serve as the key input for my construction.

Definition

Let (V, c) be a braided vector space in a braided monoidal category, for instance a Yetter–Drinfeld module over a Hopf algebra. The **Nichols algebra** $\mathcal{B}(V)$ is the graded braided Hopf algebra generated by V with relations determined by the braiding, characterized by the property that its space of primitive elements is exactly V :

$$\text{Prim}(\mathcal{B}(V)) = V.$$

Equivalently, $\mathcal{B}(V)$ is the unique (up to isomorphism) braided Hopf algebra in which V generates all primitives, and no new ones appear in higher degrees.

Definition

The **Fomin–Kirillov algebra** $\mathcal{FK}_n$ is a quadratic algebra over a field $\mathbb{k}$ generated by elements

$$\{x_{ij} \mid 1 \leq i < j \leq n\}$$

subject to the relations:

$$x_{ij}^2 = 0,$$

$$x_{ij} = -x_{ji},$$

$$x_{ij}x_{kl} = x_{kl}x_{ij} \quad \text{if } \{i, j\} \cap \{k, l\} = \emptyset,$$

$$x_{ij}x_{jk} + x_{jk}x_{ki} + x_{ki}x_{ij} = 0 \quad \text{for distinct } i, j, k.$$

The Fomin–Kirillov Algebras

These algebras have some interesting connections!

- (1) They were originally introduced to study the cohomology rings of flag varieties.
- (2) They are connected to both combinatorial and quantum algebraic structures.
- (3) Importantly, the algebra $\mathcal{FK}_3$ can be realized as a Nichols algebra of a braided vector space in the Yetter–Drinfeld category over $\mathbb{k}S_3$.

The algebra $\mathcal{FK}_3$ is generated by x_{12}, x_{13}, x_{23} subject to:

$$x_{ij}^2 = 0, \quad x_{ij}x_{jk} + x_{jk}x_{ki} + x_{ki}x_{ij} = 0.$$

We can give it a Yetter-Drinfeld structure over $\mathbb{k}S_3$ as follows:

$$g \rightharpoonup x_{ij} = \text{sgn}(g)x_{g(ij)g^{-1}}$$

$$\delta(x_{ij}) = (ij) \otimes x_{ij}$$

This algebra serves as a starting point for constructing a quantum enveloping algebra via double bosonization, and a dual quantum function algebra via codouble bosonization.

Analogy with Lie Theory

To construct a quantum group from $\mathcal{FK}_3$, we, in a sense, make a substitution into the standard decomposition of $U_q(\mathfrak{g})$

$$U_q(\mathfrak{g}) = U_q(\mathfrak{n}^-) \otimes U_q(\mathfrak{h}) \otimes U_q(\mathfrak{n}^+) \\ \uparrow \\ \mathcal{FK}_3$$

In this case, the quantum enveloping algebra associated with the Nichols algebra B of a braided vector space can be built analogously by Majid's **double bosonization**:

$$\bar{C} \rtimes H \rtimes B,$$

The double bosonization of $\mathcal{FK}_3$ was studied by Pogorelsky and Vay, but the dual quantum function algebra was not. We can compute this dual algebra through the dual process, **codouble bosonization**.

Definition

Let B be a finite-dimensional braided group in ${}^A\mathcal{M}$. Denote its dual by $B^* \in \mathcal{M}^A$. Then there is an ordinary Hopf algebra $B^{\text{op}} \rtimes A \rtimes B^*$, the co-double bosonisation, built on the vector space $B^{\text{op}} \otimes A \otimes B^*$.

Theorem

The codouble bosonization of $\mathcal{FK}_3$ as a left comodule over the dual of the Drinfeld double of $\mathbb{k}S_3$ has the following presentation in generators and relations:

Generators:

$$x_1, x_2, x_3, A, B, \{E_g\}_{g \in S_3}, y_1, y_2, y_3$$

Relations:

$$x_1^2 = x_2^2 = x_3^2 = 0 \quad x_1 x_3 = x_3 x_2 + x_2 x_1$$

$$x_3 x_1 = x_2 x_3 + x_1 x_2 \quad y_1^2 = y_2^2 = y_3^2 = 0 \quad 0 = y_3 y_2 + y_2 y_1$$

$$0 = y_2 y_3 + y_1 y_2 \quad A^2 = B^3 = 1 \quad E_g E_h = \delta_{g,h} E_g$$

$$AB = BA^{-1} \quad AE_g = E_g A \quad BE_g = E_g B \quad \sum_{g \in S_3} E_g = 1$$

Theorem Cont.

$$y_1 A = -A(E_e y_1 + E_\sigma y_1 + E_{\tau^{-1}} y_3 + E_{\tau^{-1}\sigma} y_3 + E_\tau y_2 + E_{\tau\sigma} y_2)$$

$$y_2 A = -A(E_e y_3 + E_\sigma y_3 + E_{\tau^{-1}} y_2 + E_{\tau^{-1}\sigma} y_2 + E_\tau y_1 + E_{\tau\sigma} y_1)$$

$$y_3 A = -A(E_e y_2 + E_\sigma y_2 + E_{\tau^{-1}} y_1 + E_{\tau^{-1}\sigma} y_1 + E_\tau y_3 + E_{\tau\sigma} y_3)$$

$$y_1 B = B(E_e y_3 + E_\tau y_3 + E_{\tau^{-1}} y_3 + E_\sigma y_2 + E_{\tau\sigma} y_2 + E_{\tau^{-1}\sigma} y_2)$$

$$y_2 B = B(E_e y_1 + E_\tau y_1 + E_{\tau^{-1}} y_1 + E_\sigma y_3 + E_{\tau\sigma} y_3 + E_{\tau^{-1}\sigma} y_3)$$

$$y_3 B = B(E_e y_2 + E_\tau y_2 + E_{\tau^{-1}} y_2 + E_\sigma y_1 + E_{\tau\sigma} y_1 + E_{\tau^{-1}\sigma} y_1)$$

$$y_i E_g = E_g y_i \quad A x_1 = x_1 A \quad A x_2 = x_2 B A \quad A x_3 = x_3 B^{-1} A$$

$$B x_i = x_i B^{-1} \quad y_i x_i = x_i y_i$$

So far I have constructed a new quantum function algebra via codouble bosonization. I've finished the presentation by generators and relations, but haven't finished computing the coproduct.

Future directions include:

- (1) Finishing the computation of the coproduct.
- (2) Computing the simple modules.
- (3) Computing the modular data.

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