An Exact Sequence of Hopf superalgebras Arising from the Endymion Algebra in Odd Characteristic

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Outline

Hopf Algebra Definition

Definition

A **Hopf algebra** is a bialgebra H over a field $\Bbbk$ with a $\Bbbk$-linear map $\mathcal{S} : H \to H$ called the antipode, such that the following diagram commutes:

$$
\begin{tikzcd}
 & H \otimes H \arrow[rr, "S \otimes id"] & & H \otimes H \arrow[dr, "m"] & \\
H \arrow[ur, "\Delta"] \arrow[rr, "\varepsilon"] \arrow[dr, "\Delta"'] & & \Bbbk \arrow[rr, "\eta"] & & H \\
 & H \otimes H \arrow[rr, "S \otimes id"'] & & H \otimes H \arrow[ur, "m"'] &
\end{tikzcd}
$$

Definition (Super Vector Space)

A **super vector space** is a vector space V with a C_2 grading.

$$V = \underset{\text{even}}{V_0} \oplus \underset{\text{odd}}{V_1}$$

The category of super vector spaces is symmetric, with super symmetry

$$\tau : V \otimes W \to W \otimes V, \qquad \tau(v \otimes w) = (-1)^{|v||w|} w \otimes v.$$

Hopf Superalgebras

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Definition (Hopf superalgebra)

A **Hopf superalgebra** is a Hopf algebra object within the category of super vector spaces (svec).

Exact Sequences

Definition (Exact Sequence)

A sequence of groups and homomorphisms

$$\dots \to G_{k-1} \xrightarrow{f_{k-1}} G_k \xrightarrow{f_k} G_{k+1} \to \dots$$

is called **exact** if for each k,

$$\mathrm{im}(f_{k-1}) = \ker(f_k).$$

Short Exact Sequence

Definition (Short Exact Sequence)

An exact sequence of groups of the form

$$1 \to A \overset{\iota}{\hookrightarrow} B \overset{\pi}{\twoheadrightarrow} C \to 1$$

is called a **short exact sequence**.

This means:

- The map ι is **injective**.
- The map π is **surjective**.
- $\mathrm{im}(\iota) = \ker(\pi)$

Example: The semidirect product $B \cong A \rtimes C$ gives a short exact sequence.

Braided Vector Space

Definition

A braided vector space is a pair $(\mathcal{V}, c)$ where $\mathcal{V}$ is a vector space and $c : \mathcal{V} \otimes \mathcal{V} \to \mathcal{V} \otimes \mathcal{V}$ is a linear isomorphism that satisfies the braid equation

$$(c \otimes \mathrm{id})(\mathrm{id} \otimes c)(c \otimes \mathrm{id}) = (\mathrm{id} \otimes c)(c \otimes \mathrm{id})(\mathrm{id} \otimes c).$$

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Let $q \in \Bbbk^{\times}$ and $V = \mathfrak{C}_p(q)$ be a braided vector space of dimension 3 with braiding given in the basis $\{x_i : i \in_3\}$ by

$$(c(x_i \otimes x_j))_{i,j\in\mathbb{I}_3} = \begin{pmatrix} x_1 \otimes x_1 & x_2 \otimes x_1 & qx_3 \otimes x_1 \\ x_1 \otimes x_2 & x_2 \otimes x_2 & qx_3 \otimes x_2 \\ q^{-1}x_1 \otimes x_3 & q^{-1}(x_2 + x_1) \otimes x_3 & -x_3 \otimes x_3 \end{pmatrix}.$$

Process to Construct a quantum group

- Realize (V, c) as a Hopf algebra in ${}^{\Gamma}_{\Gamma}\mathcal{YD} \Longrightarrow \mathcal{B}(V)$

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- The bosonization builds an ordinary Hopf algebra $B\#H$ from a braided Hopf algebra B in ${}^{H}_{H}\mathcal{YD}$. $B\#H = B \otimes_{\Bbbk} H$ as vector spaces and has a new multiplication and comultiplication.
    - $H := \mathcal{B}(V)\#\Bbbk\Gamma$
    - $K := H^* = \mathcal{B}(W)\#\Bbbk^{\Gamma}$

Process to Construct a quantum group

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  - $H := \mathcal{B}(V)\#\Bbbk\Gamma$
  - $K := H^* = \mathcal{B}(W)\#\Bbbk^{\Gamma}$
- The Drinfeld double $D(H)$ is the Hopf algebra defined as $D(H) = H^{*\,\mathrm{cop}} \otimes H$.

Triangular Decomposition

$D(H)$ admits a triangular decomposition

$$D(H) = D^{<0} \otimes D^{0} \otimes D^{>0},$$

where:

$$D^{<0} = B(V), \quad D^{0} = k\Gamma \otimes k^{\Gamma}, \quad D^{>0} = B(W),$$

with $B(V)$ and $B(W)$ Nichols algebras.

Exact Sequence of Hopf superalgebras

Theorem

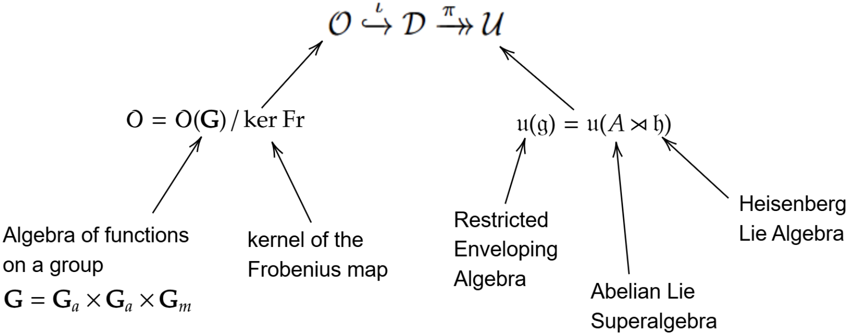
From the Drinfeld Double, we can extract a Hopf superalgebra $\mathcal{D}$. The sequence

$$\mathcal{O} \overset{\iota}{\hookrightarrow} \mathcal{D} \overset{\pi}{\twoheadrightarrow} \mathcal{U}$$

is exact where $\mathcal{O}, \mathcal{D}$, and $\mathcal{U}$ are all Hopf superalgebras.

- $\mathcal{O}$ is a commutative Hopf algebra and $\mathcal{O} \simeq \mathcal{O}(\mathfrak{G})/\ker \mathrm{Fr}$.
- $\mathcal{U}$ is the restricted enveloping algebra of a Lie superalgebra.

Exact Sequence of Hopf superalgebras (cont.)



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