The Quantum Group Associated with a Pale Block in Odd Characteristic

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Motivation

A (Drinfeld-Jimbo type) **quantum group** $U_q(\mathfrak{g})$ is a deformation of the universal enveloping algebra of a semisimple Lie algebra $\mathfrak{g}$. Why they're important:

- Generalize the symmetries of a Lie group
- Representation categories admit a braided tensor structure
- Applications to knot theory and quantum integrable systems

The positive part $U_q^+(\mathfrak{g})$ of a quantum group is a **Nichols algebra** of diagonal type over a field of characteristic 0. We can recover the quantum group structure via the **Drinfeld double** construction.

The Project

The recently discovered Nichols algebra $\mathscr{B}(\mathfrak{C}_*(q))$ is *not* of diagonal type, over a field of characteristic $p > 2$. We compute its Drinfeld Double in hopes of obtaining a generalized quantum group.

Hopf Algebras: Algebra Structure

A Hopf algebra over a field $\mathbb{k}$ has a $\mathbb{k}$-algebra structure...

Multiplication map: $\mu : H \otimes H \to H$

Unit map: $\eta : \mathbb{k} \to H$

$$
\begin{tikzcd}
H \otimes H \otimes H \arrow[r, "\mathrm{id}_H \otimes \mu"] \arrow[d, "\mu \otimes \mathrm{id}_H"'] & H \otimes H \arrow[d, "\mu"] \\
H \otimes H \arrow[r, "\mu"'] & H
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbb{k} \otimes H \arrow[r, "\eta \otimes \mathrm{id}_H"] \arrow[dr, "="'] & H \otimes H \arrow[d, "\mu"'] & H \otimes \mathbb{k} \arrow[l, "\mathrm{id}_H \otimes \eta"'] \arrow[dl, "="] \\
& H &
\end{tikzcd}
$$

These commutative diagrams represent associativity and unitality:

$$\mu(\mu(x \otimes y) \otimes z)) = \mu(x \otimes \mu(y \otimes z)),$$
$$\eta(1_{\mathbb{k}}) = 1_H.$$

Hopf Algebras: (Co)Algebra Structure

...and a $\Bbbk$-coalgebra structure:

Comultiplication map:	$\Delta : H \to H \otimes H$
Counit map:	$\varepsilon : H \to \Bbbk$

$$\begin{tikzcd} H \otimes H \otimes H & H \otimes H \arrow[l, "\Delta \otimes \mathrm{id}_H"'] \\ H \otimes H \arrow[u, "\mathrm{id}_H \otimes \Delta"] & H \arrow[l, "\Delta"] \arrow[u, "\Delta"'] \end{tikzcd} \qquad \begin{tikzcd} H \otimes \Bbbk & H \otimes H \arrow[l, "\mathrm{id}_H \otimes \varepsilon"'] \arrow[r, "\varepsilon \otimes \mathrm{id}_H"] & \Bbbk \otimes H \\ & H \arrow[ul, "="] \arrow[u, "\Delta"'] \arrow[ur, "="'] & \end{tikzcd}$$

We shorthand $\Delta(x) = \sum x_i \otimes x_j$ to $x_{(1)} \otimes x_{(2)}$:

$$\Delta(x_{(1)}) \otimes x_{(2)} = x_{(1)} \otimes \Delta(x_{(2)}) := x_{(1)} \otimes x_{(2)} \otimes x_{(3)},$$
$$\varepsilon(x_{(1)})x_{(2)} = x_{(1)}\varepsilon(x_{(2)}) = x.$$

The algebra and coalgebra structures are **compatible**; that is, for $x, y \in H$,

$$(xy)_{(1)} \otimes (xy)_{(2)} = x_{(1)}y_{(1)} \otimes x_{(2)}y_{(2)}.$$

Hopf Algebras

In addition, elements in a Hopf algebra have an **antipode** $\mathcal{S} : H \to H$, which is a generalized notion of the inverse:

$$\mathcal{S}(x_{(1)})x_{(2)} = \varepsilon(x)1_H = x_{(1)}\mathcal{S}(x_{(2)}).$$

Example

The group algebra $\Bbbk G$ is the simplest example of a Hopf algebra. (Indeed, the Hopf algebra is a generalization of the group algebra!)

$$\begin{aligned} \mu(g \otimes h) &= gh \\ \Delta(g) &= g \otimes g \\ \mathcal{S}(g) &= g^{-1} \end{aligned}$$

Braided Hopf Algebras

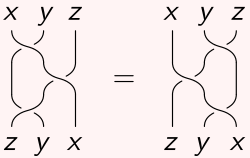
Definition

A *braiding* is a map $c : H \otimes H \to H \otimes H$ that satisfies

$$(c \otimes \mathrm{id}_H)(\mathrm{id}_H \otimes c)(c \otimes \mathrm{id}_H) = (\mathrm{id}_H \otimes c)(c \otimes \mathrm{id}_H)(\mathrm{id}_H \otimes c)$$

Example

The flip map $\tau : x \otimes y \mapsto y \otimes x$ is a braiding:



In a **braided Hopf algebra**, the μ-Δ relation is modified by a braiding c.

Let V be a vector space, H a Hopf algebra. V has an H-**(co)module** structure if:

Action:	$\lambda : H \otimes V \to V$
Coaction:	$\delta : V \to H \otimes V$

$$\begin{tikzcd}
H \otimes H \otimes V \arrow[rr, shift left, "\mu \otimes \mathrm{id}_V"] \arrow[d, shift right, "\mathrm{id}_A \otimes \lambda"'] & & H \otimes V \arrow[ll, shift left, "\Delta \otimes \mathrm{id}_V"] \arrow[d, shift left, "\lambda"] \\
H \otimes V \arrow[u, shift right, "\mathrm{id}_{\Bbbk G} \otimes \delta"'] \arrow[rr, shift right, "\lambda"'] & & V \arrow[ll, shift right, "\delta"'] \arrow[u, shift left, "\delta"]
\end{tikzcd}
\qquad
\begin{tikzcd}
\Bbbk \otimes V \arrow[r, shift left, "\eta \otimes \mathrm{id}_V"] \arrow[dr, leftrightarrow, "="] & H \otimes V \arrow[l, shift left, "\varepsilon \otimes \mathrm{id}_V"] \arrow[d, shift left, "\lambda"] \\
 & V \arrow[u, shift left, "\delta"]
\end{tikzcd}$$

We write $\delta(v) = v_{(-1)} \otimes v_{(0)}$ as shorthand, so we have relations:

$$\Delta(v_{(-1)}) \otimes v_{(0)} = v_{(-1)} \otimes \delta(v_{(0)}) := v_{(-2)} \otimes v_{(-1)} \otimes v_{(0)},$$
$$\varepsilon(v_{(-1)})v_{(0)} = v.$$

Recovering a Quantum Group

The Nichols algebra $\mathscr{B}(V)$ is a type of braided Hopf algebra constructed from V, a $\Bbbk\Gamma$ (co)module, where Γ is an abelian group.

$$V \xrightarrow{\text{Nichols alg.}} \mathscr{B}(V) \xrightarrow{\text{bosonization}} \mathscr{B}(V) \# \Bbbk\Gamma \xrightarrow{\text{Drinfeld double}} D(\mathscr{B}(V) \# \Bbbk\Gamma)$$

- The **bosonization** $\mathscr{B}(V) \# \Bbbk\Gamma$ has underlying vector space structure $\mathscr{B}(V) \otimes \Bbbk\Gamma$, and is a (non-braided) Hopf algebra.
- The **Drinfeld double** $D(\mathscr{B}(V) \# \Bbbk\Gamma)$ has underlying vector space structure

$$\begin{aligned}(\mathscr{B}(V) \# \Bbbk\Gamma) \otimes (\mathscr{B}(V) \# \Bbbk\Gamma)^* &= \mathscr{B}(V) \otimes \Bbbk\Gamma \otimes \mathscr{B}(V^*) \otimes \Bbbk\Gamma^* \\ &= \underbrace{\mathscr{B}(V)}_{U_q^+(\mathfrak{g})} \otimes \underbrace{(\Bbbk\Gamma \otimes \Bbbk^\Gamma)}_{U_q^0(\mathfrak{g})} \otimes \underbrace{\mathscr{B}(V^*)}_{U_q^-(\mathfrak{g})}.\end{aligned}$$

The Nichols algebras associated with quantum groups via this construction are of **diagonal type** and over a field of characteristic 0.

The vector space $\mathfrak{C}_*(q)$

We now work in a field $\Bbbk$ of characteristic $p > 2$. Let q be an Nth root of unity. The vector space $\mathfrak{C}_*(q)$ is a $\Bbbk\Gamma$ (co)module, where $\Gamma = C_{2N} \times C_{2pN}$. It is generated by v_1, v_2 (the pale block), and v_3 (the point).

Group actions:

$$\begin{aligned} g_1 \rightharpoonup v_1 &= -v_1, & g_2 \rightharpoonup v_1 &= -q^{-1} v_1, \\ g_1 \rightharpoonup v_2 &= -v_2, & g_2 \rightharpoonup v_2 &= -q^{-1}(v_2 + v_1), \\ g_1 \rightharpoonup v_3 &= q v_3, & g_2 \rightharpoonup v_3 &= -v_3. \end{aligned}$$

Group coactions:

$$\delta(v_1) = g_1 \otimes v_1, \qquad \delta(v_2) = g_1 \otimes v_2, \qquad \delta(v_3) = g_2 \otimes v_3.$$

This induces the braiding, **not of diagonal type**:

$$\begin{aligned} c(v_1 \otimes v_1) &= -v_1 \otimes v_1, & c(v_1 \otimes v_2) &= -v_2 \otimes v_1, & c(v_1 \otimes v_3) &= q v_3 \otimes v_1, \\ c(v_2 \otimes v_1) &= -v_1 \otimes v_2, & c(v_2 \otimes v_2) &= -v_2 \otimes v_2, & c(v_2 \otimes v_3) &= q v_3 \otimes v_2, \\ c(v_3 \otimes v_1) &= -q^{-1} v_1 \otimes v_3, & c(v_3 \otimes v_2) &= -q^{-1}(v_2 + v_1) \otimes v_3, & c(v_3 \otimes v_3) &= -v_3 \otimes v_3. \end{aligned}$$

The Drinfeld double $D(\mathscr{B}(\mathfrak{C}_*(q))\,\#\,\Bbbk\Gamma)$

Recall: the double has vector space structure $\mathscr{B}(V)\otimes\Bbbk\Gamma\otimes\mathscr{B}(V^*)\otimes\Bbbk^\Gamma$.

$$\mathscr{B}(V)\to\langle v_1,v_2,v_3\rangle \qquad \Bbbk\Gamma\to\langle\alpha_1,\alpha_2,\alpha_3\rangle$$

$$\mathscr{B}(V^*)\to\langle u_1,u_2,u_3\rangle \qquad \Bbbk^\Gamma\to\langle\tau_1,\tau_2,\tau_3\rangle$$

Some relations (between v_i and u_i):

- $u_1v_1=-v_1u_1$,
- $u_1v_2=1-v_2u_1-\alpha_1\tau_1^{-N}\tau_2^{-2p-N}$,
- $u_1v_3=-q^{-1}v_3u_1$,
- $u_2v_1=1-v_1u_2-\alpha_1\tau_1^{-N}\tau_2^{-2p-N}$,
- $u_2v_2=-v_2u_2-2N\alpha_1\tau_1^{-N}\tau_2^{-2p-N}\tau_3$,
- $u_2v_3=-q^{-1}(v_3u_2+v_3u_1)$,
- $u_3v_1=qv_1u_3$,
- $u_3v_2=qv_2u_3$,
- $u_3v_3=1-v_3u_3-g_2\tau_1^2\tau_2^{-N}$.

Future Directions

Due to the exotic nature of the double, characterizing its structure is difficult. A few paths to go down:

- Express double as a superalgebra
- Relating to known algebras:
  - Find exact sequence $\mathcal{A} \hookrightarrow \mathcal{D} \twoheadrightarrow \mathcal{C}$
  - Relation to $\mathfrak{osp}(1|2)$
- Compute simple modules over $\mathcal{D}$
  - New knot invariants

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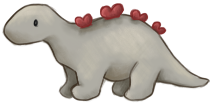

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