On a Quantum Group in Characteristic 2

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Algebras

A k-algebra is a k-vector space A with a linear multiplication map $\mu: A \otimes A \to A$ and a linear unit map $\eta: k \to A$ satisfying

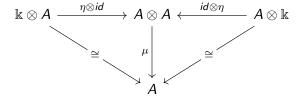
$$\begin{array}{ccc}
A \otimes A \otimes A & \xrightarrow{\mu \otimes id} & A \otimes A \\
\downarrow^{id \otimes \mu} & & \downarrow^{\mu} \\
A \otimes A & \xrightarrow{\mu} & A
\end{array}$$

i.e classic associativity: (xy)z = x(yz) for all $x, y, z \in A$.

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Algebras

It also satisfies



i.e.
$$\eta(1)x = x = x\eta(1)$$
 for all $x \in A$.

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Modules

A left A-module is a vector space V with a linear action map $\lambda: A \otimes V \to V$ satisfying

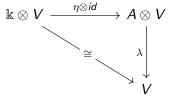
$$\begin{array}{ccc}
A \otimes A \otimes V & \xrightarrow{\mu \otimes id_{V}} & A \otimes V \\
\downarrow^{id_{A} \otimes \lambda} & & \downarrow^{\lambda} \\
A \otimes V & \xrightarrow{\lambda} & V
\end{array}$$

If we denote $\lambda(a \otimes v) = a \cdot v$, this says that $(xy) \cdot v = x \cdot (y \cdot v)$ for all $x, y \in A$ and $v \in V$.

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Modules

It also satisfies

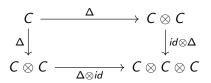


i.e. $\eta(1) \cdot v = v$ for all $v \in V$.

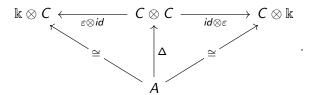
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Coalgebras

Flip the arrows to define a &-coalgebra C, or a &-vector space C with a linear comultiplication map $\Delta:C\to C\otimes C$ and a linear counit map $\varepsilon:C\to \&$ satisfying



and



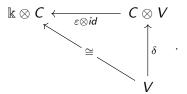
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Comodules

A left C-comodule is a vector space V with a linear coaction map $\delta: V \to C \otimes V$ satisfying:

$$\begin{array}{ccc}
V & \xrightarrow{\delta} & C \otimes V \\
\downarrow \delta \downarrow & & \downarrow \Delta \otimes id \\
C \otimes V & \xrightarrow{id \otimes \Delta} & C \otimes C \otimes V
\end{array}$$

and



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Example: The Group Algebra

The group algebra kG has

- an algebra structure: group multiplication and the group unit.
- a coalgebra structure: $\Delta(g) = g \otimes g$ and $\varepsilon(g) = 1$ for all $g \in G$.
- a bialgebra structure: $\Delta(xy) = \Delta(x)\Delta(y)$ and $\varepsilon(xy) = \varepsilon(x)\varepsilon(y)$.
- an inverse: the group inverse.

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Hopf Algebras

A Hopf algebra H is a bialgebra with a linear antipode map $S: H \to H$ satisfying:

$$h_{(1)}S(h_{(2)}) = \varepsilon(h)\eta(1) = S(h_{(1)})h_{(2)}$$

where $\Delta(h) = h_{(1)} \otimes h_{(2)}$.

The most natural symmetry induced by the tensor product is the flip map $\tau_{X,Y}: X \otimes Y \to Y \otimes X$ taking $x \otimes y$ to $y \otimes x$.

- $\tau_{H,H'}$ is a Hopf algebra map for H and H' Hopf algebras
- $\tau_{V,W}$ is not necessarily a (co)module map for V and W (co)modules

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Yetter-Drinfeld Modules

A Yetter-Drinfeld module V over a Hopf algebra H is a vector space with a module and comodule structure $\lambda: H \otimes V \to V$ and $\delta: V \to H \otimes V$, $v \mapsto v_{(-1)} \otimes v_{(0)}$ satisfying

$$\delta(h \cdot v) = h_{(1)}v_{(-1)}S(h_{(3)}) \otimes h_{(2)} \cdot v_{(0)}.$$

For $V, W \in {}_{H}^{H}\mathcal{YD}$, the map $c_{V,W} : V \otimes W \to W \otimes V$ with $c_{V,W}(v \otimes w) = v_{(-1)} \cdot w \otimes v_{(0)}$ is both a module and a comodule map.

- If $c_{V,W} = \tau_{V,W}$: classical symmetry
- If $c_{V,W} \neq \tau_{V,W}$: quantum symmetry

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Primitives and Infinitesimal Symmetry

For H a Hopf algebra, we say $h \in H$ is primitive if $\Delta(h) = 1 \otimes h + h \otimes 1$. Then, for V and W modules over H,

$$h \cdot (v \otimes w) = h \cdot v \otimes w + v \otimes h \cdot w$$

 \dots h acts like a derivative (it's the algebraic analog of infinitesimal symmetry).

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The Quantum Case - More Constructions

• For $V \in {}_{H}^{H}\mathcal{YD}$, the Nichols Algebra is the "smallest" object in ${}_{H}^{H}\mathcal{YD}$ generated by its primitives V:

$$B(V) := T(V)/I(V)$$

where $T(V) = \{v_1v_2...v_n : v_i \in V, n \in \mathbb{N}\}$ is the tensor algebra.

• The Drinfeld Double twists H and H^* to get a quantum group with

$$D(H)$$
-mod $\simeq {}_{H}^{H}\mathcal{Y}\mathcal{D}$.

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Our Project - Big Picture

Yetter-Drinfeld Module in Characteristic 2

Nichols Algebra

Hopf Algebra

Drinfeld Double

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Our Project - Yetter-Drinfeld Module

In 2021, Andruskiewitsch, Angiono, and Giusti introduced $\mathfrak{E}_{3,-}(q) \in \frac{\Bbbk(C_m \times C_{4m})}{\Bbbk(C_m \times C_{4m})} \mathcal{YD}$ with (co)action given by:

$$g \cdot x_4 = qx_4,$$
 $g \cdot x_i = x_i \quad \forall 1 \le i \le 3,$ $h \cdot x_1 = q^{-1}x_1,$ $h \cdot x_2 = q^{-1}(x_1 + x_2),$ $h \cdot x_3 = q^{-1}(x_2 + x_3),$ $h \cdot x_4 = x_4,$ $\delta(x_4) = h \otimes x_4,$ $\delta(x_i) = g \otimes x_i \quad \forall 1 < i < 3.$

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Our Results - The Nichols Algebra

 $\mathfrak{B}\left(\mathfrak{E}_{3,-}(q)\right)$ is generated by x_1,x_2,x_3,x_4 with the relations

$$x_i^2 = 0,$$
 $x_i x_j = x_j x_i \quad \forall 1 \le i \ne j \le 3,$ $x_4^2 = 0,$ $x_1 x_4 = q x_4 x_1,$ $x_2^2 = 0,$ $x_3^2 x_2 + x_2 x_3^2 + x_2 x_3 x_2 = 0,$ $x_3 x_2 x_3 x_2 + x_2 x_3 x_2 x_3 = 0$

where

$$z_i = x_4 x_i + q^{-1} (x_i + x_{i-1}) x_4,$$

 $w = z_2 x_3 + q^{-1} (x_3 + x_2) z_2,$
 $y = z_2 z_3 + z_3 z_2.$

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Our Results - The Hopf Algebra

 $H = \mathfrak{B}\left(\mathfrak{E}_{3,-}(q)\right) \# \mathbb{k}(C_m \times C_{4m})$ is generated by x_1, x_2, x_3, x_4, g and h with the relations

$$x_i^2 = 0,$$
 $x_i x_j = x_j x_i$ $\forall 1 \le i \ne j \le 3,$
 $x_4^2 = 0,$ $x_1 x_4 = q x_4 x_1,$
 $z_2^2 = 0,$ $z_3^4 = 0,$
 $z_3^2 z_2 + z_2 z_3^2 + z_2 z_3 z_2 = 0,$ $z_3 z_2 z_3 z_2 + z_2 z_3 z_2 z_3 = 0,$
 $g^m = h^{4m} = 1,$ $gh = hg,$
 $gx_4 = qx_4 g,$ $x_i g = gx_i$ $\forall 1 \le i \le 3,$
 $hx_1 = q^{-1} x_1 h,$ $hx_2 = q^{-1} (x_1 + x_2) h,$
 $hx_3 = q^{-1} (x_2 + x_3) h,$ $hx_4 = x_4 h.$

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Our Results - The Drinfeld Double

- D(H) has 15 generators and over 50 relations.
- Since D(H)-mod $\simeq {}_{H}^{H}\mathcal{YD}$, we are particularly interested in the modules over D(H). We have explicit presentations of some of the simples, i.e. (for $\zeta_n = e^{\frac{2\pi i}{n}}$):

$$\begin{array}{lll} g \cdot 1 = \zeta_m^{e_g}, & g \cdot w_1 = \zeta_m^{e_g} w_1, \\ h \cdot 1 = \zeta_m^{-e_u}, & h \cdot w_1 = q \zeta_m^{-e_u} w_1, \\ v \cdot 1 = \zeta_m^{-e_g}, & v \cdot w_1 = \zeta_m^{-e_g} w_1, \\ u \cdot 1 = \zeta_m^{e_u}, & u \cdot w_1 = q^{-1} \zeta_m^{e_u} w_1, \\ s \cdot 1 = 0, & s \cdot w_1 = 0, \\ t \cdot 1 = 1, & t \cdot w_1 = 1, \\ x_i \cdot 1 = 0 & \forall i \neq 3, & x_i \cdot w_1 = 0 & \forall i \neq 3, \\ x_3 \cdot 1 = 0, & x_3 \cdot w_1 = 1, \\ w_i \cdot 1 = 0 & \forall i \neq 1, & w_i \cdot w_1 = 0 & \forall 1 \leq i \leq 4. \end{array}$$

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