# On a Quantum Group in Characteristic 2

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# Algebras

A $\mathbb{k}$-algebra is a $\mathbb{k}$-vector space $A$ with a linear multiplication map $\mu : A \otimes A \to A$ and a linear unit map $\eta : \mathbb{k} \to A$ satisfying

$$\begin{tikzcd}
A \otimes A \otimes A \arrow[r, "\mu\otimes id"] \arrow[d, "id\otimes\mu"'] & A \otimes A \arrow[d, "\mu"] \\
A \otimes A \arrow[r, "\mu"'] & A
\end{tikzcd}$$

i.e classic associativity: $(xy)z = x(yz)$ for all $x, y, z \in A$.

# Algebras

It also satisfies

$$\begin{tikzcd}
\Bbbk \otimes A \arrow[r, "\eta \otimes id"] \arrow[dr, "\cong"'] & A \otimes A \arrow[d, "\mu"'] & A \otimes \Bbbk \arrow[l, "id \otimes \eta"'] \arrow[dl, "\cong"] \\
& A &
\end{tikzcd}$$

i.e. $\eta(1)x = x = x\eta(1)$ for all $x \in A$.

A left A-module is a vector space $V$ with a linear action map $\lambda : A \otimes V \to V$ satisfying

$$\begin{tikzcd}
A \otimes A \otimes V \arrow[r, "\mu \otimes id_V"] \arrow[d, "id_A \otimes \lambda"'] & A \otimes V \arrow[d, "\lambda"] \\
A \otimes V \arrow[r, "\lambda"'] & V
\end{tikzcd}$$

If we denote $\lambda(a \otimes v) = a \cdot v$, this says that $(xy) \cdot v = x \cdot (y \cdot v)$ for all $x, y \in A$ and $v \in V$.

# Modules

It also satisfies

$$\begin{tikzcd}
\Bbbk \otimes V \arrow[r, "\eta \otimes id"] \arrow[dr, "\cong"'] & A \otimes V \arrow[d, "\lambda"] \\
& V
\end{tikzcd}$$

i.e. $\eta(1) \cdot v = v$ for all $v \in V$.

# Coalgebras

Flip the arrows to define a $\Bbbk$-coalgebra $C$, or a $\Bbbk$-vector space $C$ with a linear comultiplication map $\Delta : C \to C \otimes C$ and a linear counit map $\varepsilon : C \to \Bbbk$ satisfying

$$\begin{tikzcd}
C \arrow[r, "\Delta"] \arrow[d, "\Delta"'] & C \otimes C \arrow[d, "id \otimes \Delta"] \\
C \otimes C \arrow[r, "\Delta \otimes id"'] & C \otimes C \otimes C
\end{tikzcd}$$

and

$$\begin{tikzcd}
\Bbbk \otimes C & C \otimes C \arrow[l, "\varepsilon \otimes id"'] \arrow[r, "id \otimes \varepsilon"'] & C \otimes \Bbbk \\
& A \arrow[ul, "\cong"] \arrow[u, "\Delta"'] \arrow[ur, "\cong"'] &
\end{tikzcd}.$$

A left C-comodule is a vector space $V$ with a linear coaction map $\delta : V \to C \otimes V$ satisfying:

$$\begin{tikzcd} V \arrow[r, "\delta"] \arrow[d, "\delta"'] & C \otimes V \arrow[d, "\Delta \otimes id"] \\ C \otimes V \arrow[r, "id \otimes \Delta"'] & C \otimes C \otimes V \end{tikzcd}$$

and

$$\begin{tikzcd} \Bbbk \otimes C & C \otimes V \arrow[l, "\varepsilon \otimes id"'] \\ & V \arrow[u, "\delta"'] \arrow[ul, "\cong"] \end{tikzcd} .$$

# Example: The Group Algebra

The group algebra $\Bbbk G$ has

- an algebra structure: group multiplication and the group unit.
- a coalgebra structure: $\Delta(g) = g \otimes g$ and $\varepsilon(g) = 1$ for all $g \in G$.
- a bialgebra structure: $\Delta(xy) = \Delta(x)\Delta(y)$ and $\varepsilon(xy) = \varepsilon(x)\varepsilon(y)$.
- an inverse: the group inverse.

# Hopf Algebras

A Hopf algebra $H$ is a bialgebra with a linear antipode map $S : H \to H$ satisfying:

$$h_{(1)}S(h_{(2)}) = \varepsilon(h)\eta(1) = S(h_{(1)})h_{(2)}$$

where $\Delta(h) = h_{(1)} \otimes h_{(2)}$.

The most natural symmetry induced by the tensor product is the flip map $\tau_{X,Y} : X \otimes Y \to Y \otimes X$ taking $x \otimes y$ to $y \otimes x$.

- $\tau_{H,H'}$ is a Hopf algebra map for $H$ and $H'$ Hopf algebras
- $\tau_{V,W}$ is not necessarily a (co)module map for $V$ and $W$ (co)modules

# Yetter-Drinfeld Modules

A Yetter-Drinfeld module $V$ over a Hopf algebra $H$ is a vector space with a module and comodule structure $\lambda : H \otimes V \to V$ and $\delta : V \to H \otimes V$, $v \mapsto v_{(-1)} \otimes v_{(0)}$ satisfying

$$\delta(h \cdot v) = h_{(1)} v_{(-1)} S(h_{(3)}) \otimes h_{(2)} \cdot v_{(0)}.$$

For $V, W \in {}^H_H\mathcal{YD}$, the map $c_{V,W} : V \otimes W \to W \otimes V$ with $c_{V,W}(v \otimes w) = v_{(-1)} \cdot w \otimes v_{(0)}$ is both a module and a comodule map.

- If $c_{V,W} = \tau_{V,W}$ : classical symmetry
- If $c_{V,W} \neq \tau_{V,W}$ : quantum symmetry

# Primitives and Infinitesimal Symmetry

For $H$ a Hopf algebra, we say $h \in H$ is primitive if $\Delta(h) = 1 \otimes h + h \otimes 1$. Then, for $V$ and $W$ modules over $H$,

$$h \cdot (v \otimes w) = h \cdot v \otimes w + v \otimes h \cdot w$$

... $h$ acts like a derivative (it's the algebraic analog of infinitesimal symmetry).

# The Quantum Case - More Constructions

- For $V \in {}^H_H\mathcal{YD}$, the Nichols Algebra is the "smallest" object in ${}^H_H\mathcal{YD}$ generated by its primitives $V$:

$$B(V) := T(V)/I(V)$$

where $T(V) = \{v_1 v_2 \ldots v_n : v_i \in V, n \in \mathbb{N}\}$ is the tensor algebra.

- The Drinfeld Double twists $H$ and $H^*$ to get a quantum group with

$$D(H)\text{-mod} \simeq {}^H_H\mathcal{YD}.$$

# Our Project - Big Picture

Yetter-Drinfeld Module in Characteristic 2

$\downarrow$

Nichols Algebra

$\downarrow$

Hopf Algebra

$\downarrow$

Drinfeld Double

# Our Project - Yetter-Drinfeld Module

In 2021, Andruskiewitsch, Angiono, and Giusti introduced $\mathfrak{E}_{3,-}(q) \in {}^{\Bbbk(C_m \times C_{4m})}_{\Bbbk(C_m \times C_{4m})}\mathcal{YD}$ with (co)action given by:

$$
\begin{aligned}
g \cdot x_4 &= q x_4, & g \cdot x_i &= x_i \quad \forall 1 \leq i \leq 3, \\
h \cdot x_1 &= q^{-1} x_1, & h \cdot x_2 &= q^{-1}(x_1 + x_2), \\
h \cdot x_3 &= q^{-1}(x_2 + x_3), & h \cdot x_4 &= x_4, \\
\delta(x_4) &= h \otimes x_4, & \delta(x_i) &= g \otimes x_i \quad \forall 1 \leq i \leq 3.
\end{aligned}
$$

# Our Results - The Nichols Algebra

$\mathfrak{B}\left(\mathfrak{E}_{3,-}(q)\right)$ is generated by $x_1, x_2, x_3, x_4$ with the relations

$$
\begin{aligned}
x_i^2 &= 0, & x_i x_j &= x_j x_i \quad \forall 1 \leq i \neq j \leq 3,\\
x_4^2 &= 0, & x_1 x_4 &= q x_4 x_1,\\
z_2^2 &= 0, & z_3^4 &= 0,\\
z_3^2 z_2 + z_2 z_3^2 + z_2 z_3 z_2 &= 0, & z_3 z_2 z_3 z_2 + z_2 z_3 z_2 z_3 &= 0
\end{aligned}
$$

where

$$
\begin{aligned}
z_i &= x_4 x_i + q^{-1}(x_i + x_{i-1}) x_4,\\
w &= z_2 x_3 + q^{-1}(x_3 + x_2) z_2,\\
y &= z_2 z_3 + z_3 z_2.
\end{aligned}
$$

$H = \mathfrak{B}\left(\mathfrak{E}_{3,-}(q)\right) \# \Bbbk(C_m \times C_{4m})$ is generated by $x_1, x_2, x_3, x_4, g$ and $h$ with the relations

$$
\begin{aligned}
x_i^2 &= 0, & x_i x_j &= x_j x_i \quad \forall 1 \leq i \neq j \leq 3, \\
x_4^2 &= 0, & x_1 x_4 &= q x_4 x_1, \\
z_2^2 &= 0, & z_3^4 &= 0, \\
z_3^2 z_2 + z_2 z_3^2 + z_2 z_3 z_2 &= 0, & z_3 z_2 z_3 z_2 + z_2 z_3 z_2 z_3 &= 0, \\
g^m = h^{4m} &= 1, & gh &= hg, \\
g x_4 &= q x_4 g, & x_i g &= g x_i \quad \forall 1 \leq i \leq 3, \\
h x_1 &= q^{-1} x_1 h, & h x_2 &= q^{-1}(x_1 + x_2) h, \\
h x_3 &= q^{-1}(x_2 + x_3) h, & h x_4 &= x_4 h.
\end{aligned}
$$

# Our Results - The Drinfeld Double

- $D(H)$ has 15 generators and over 50 relations.
- Since $D(H)$-mod $\simeq {}^H_H\mathcal{YD}$, we are particularly interested in the modules over $D(H)$. We have explicit presentations of some of the simples, i.e. (for $\zeta_n = e^{\frac{2\pi i}{n}}$):

$$
\begin{aligned}
g \cdot 1 &= \zeta_m^{e_g}, & g \cdot w_1 &= \zeta_m^{e_g} w_1, \\
h \cdot 1 &= \zeta_m^{-e_u}, & h \cdot w_1 &= q\zeta_m^{-e_u} w_1, \\
v \cdot 1 &= \zeta_m^{-e_g}, & v \cdot w_1 &= \zeta_m^{-e_g} w_1, \\
u \cdot 1 &= \zeta_m^{e_u}, & u \cdot w_1 &= q^{-1}\zeta_m^{e_u} w_1, \\
s \cdot 1 &= 0, & s \cdot w_1 &= 0, \\
t \cdot 1 &= 1, & t \cdot w_1 &= 1, \\
x_i \cdot 1 &= 0 \quad \forall i \neq 3, & x_i \cdot w_1 &= 0 \quad \forall i \neq 3, \\
x_3 \cdot 1 &= 0, & x_3 \cdot w_1 &= 1, \\
w_i \cdot 1 &= 0 \quad \forall i \neq 1, & w_i \cdot w_1 &= 0 \quad \forall 1 \leq i \leq 4.
\end{aligned}
$$

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