On a Curious Observation on the Jones Polynomial, And How to Categorify It

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Knots and Links

Definitions

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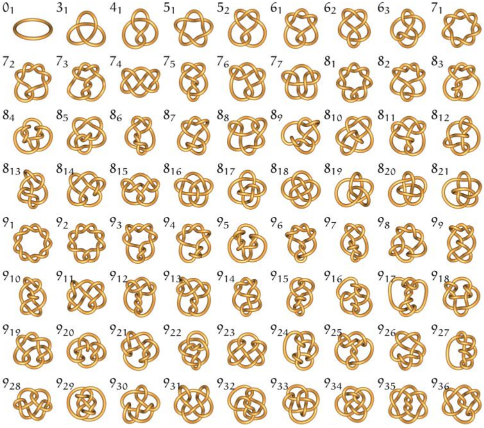
Link diagrams project knots and links onto two-dimensional space.



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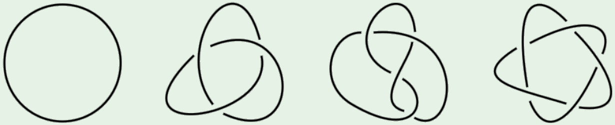

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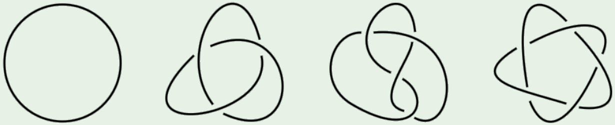

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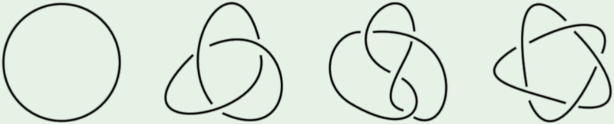

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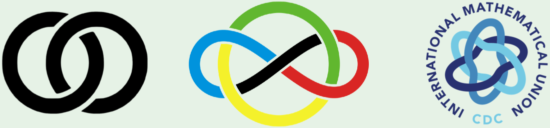

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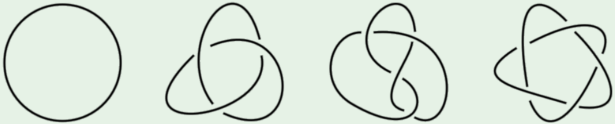

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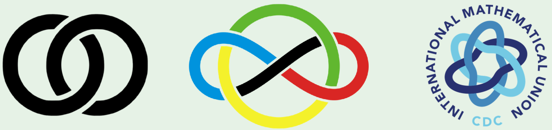

Hopf link (2^2_1), **Whitehead link (5^2_1)**, and **Borromean link (6^3_2)**.

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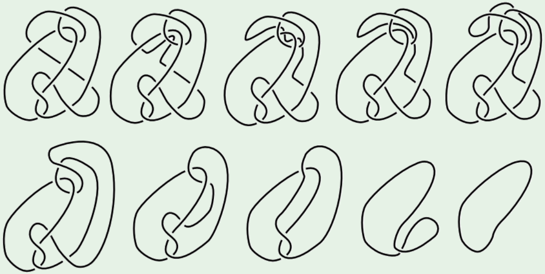

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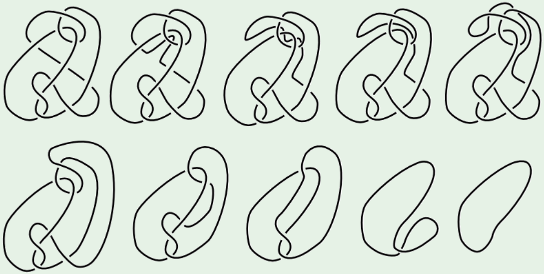

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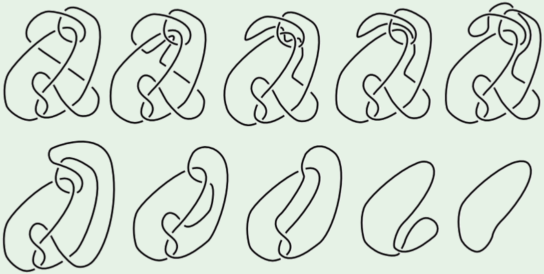

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Try to simplify the (topological) notion of isotopy to be more tractable.

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Theorem (Jones, 1985)

The Jones polynomial is a well-defined polynomial of (oriented) links. Furthermore, it is **invariant** under isotopy.

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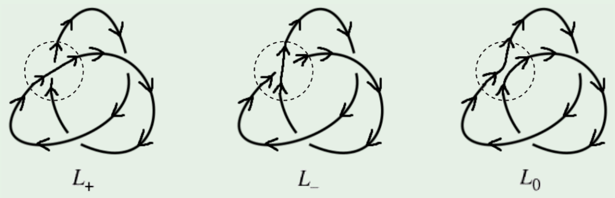

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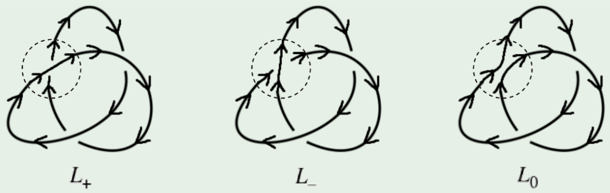

Observe that $(L_+, L_-, L_0) = (0_1, 3_1, 2_1^2)$. Thus

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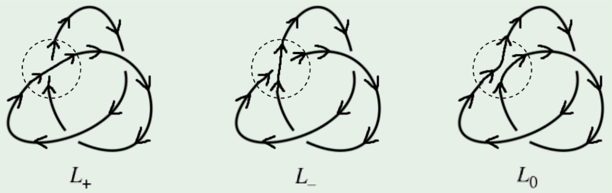

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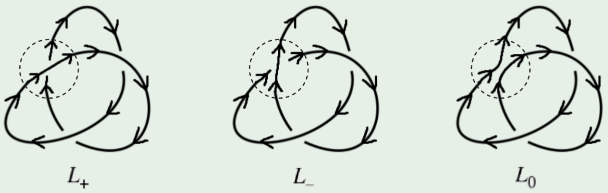

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Jones originally constructed the polynomial using representation theory.

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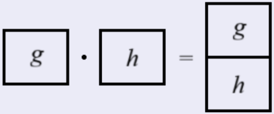

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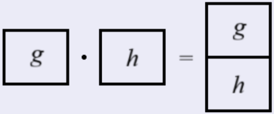

- All closed loops may be factored out as β.

The Temperley–Lieb Algebra

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- The dimension of $\mathrm{TL}_3(\beta)$ is 5:

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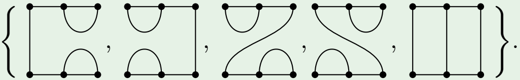

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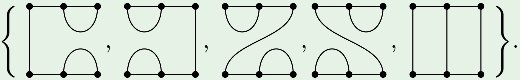
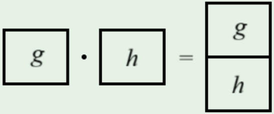
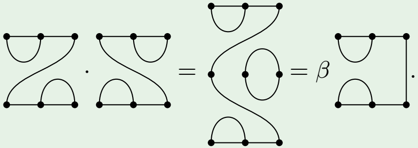
$\mathrm{TL}_3(\beta) = \mathrm{span}$ 

- A demonstration of diagram concatenation:





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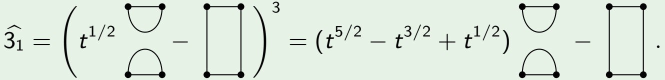

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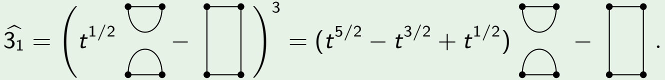
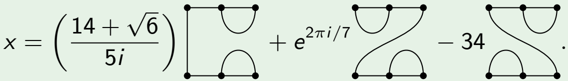


Remark

This demonstrates why you should not think of $\mathrm{TL}_n(\beta)$ as "just" a collection of diagrams. Another example: a generic element of $\mathrm{TL}_3(\beta)$ might look like



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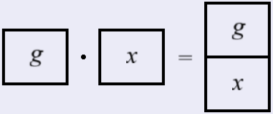

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$$\boxed{g} \cdot \boxed{x} = \begin{array}{|c|}\hline g \\ \hline x \\ \hline \end{array}$$

Definition

- The **trace** $\mathrm{tr}(A)$ of a square matrix A sums its diagonal entries.
- The **character** $\chi_\ell^n \colon \mathrm{TL}_n(\beta) \to \mathbb{C}$ of the standard module W_ℓ^n is defined as $\chi_\ell^n(g) = \mathrm{tr}_{W_\ell^n}(g)$. (Each $g \in \mathrm{TL}_n(\beta)$ acts as a linear transformation on the standard module W_ℓ^n.)

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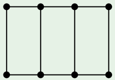
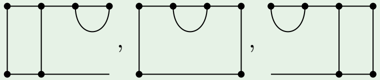
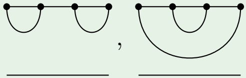
$$W_4^4 = \mathrm{span}\left\{ \right.$$  $$\left. \right\}$$

$$W_2^4 = \mathrm{span}\left\{ \right.$$  $$\left. \right\}$$

$$W_0^4 = \mathrm{span}\left\{ \right.$$  $$\left. \right\}$$

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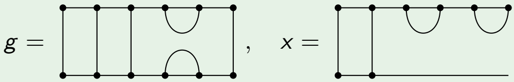

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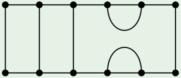
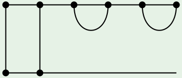
$g =$  , $x =$ 

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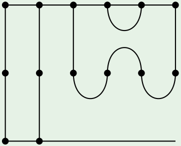
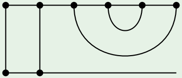
$gx =$  $=$ 

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Fix nonzero $t \in \mathbb{C}$, and let $\beta = t^{1/2} + t^{-1/2}$. Recall that:

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Proposition (Jones, 1987)

The Jones polynomial of an oriented link L is given by

$$V_L(t) = \pm \frac{t^{c/2}}{1+t} \sum_{\substack{\ell=0, \\ 2|(\ell+n)}}^{n} \left(\sum_{i=(n-\ell)/2}^{(n+\ell)/2} t^i \right) \chi_\ell^n(\hat{L})$$

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- If n is odd, $\sum_{i=(n-\ell)/2}^{(n+\ell)/2} (-1)^i = 0$, so everything is fine.
- If n is even, $\sum_{i=(n-\ell)/2}^{(n+\ell)/2} (-1)^i = \pm 1$... something strange is going on!

Alternating Sum

When n is even, at $t = -1$ we obtain an identity for $\beta = 0$:

$$\chi_0^n(\hat{L}) - \chi_2^n(\hat{L}) + \chi_4^n(\hat{L}) - \cdots \pm \chi_n^n(\hat{L}) = \sum_{\substack{\ell=0,\\ \ell \text{ even}}}^{n} (-1)^{\ell/2} \chi_\ell^n(\hat{L}) = 0.$$

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Overarching Question

Is there a more conceptual way to make sense of the above sum?

Short Exact Sequences

A little tangent.

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Consider vector spaces U, V, and W equipped with linear maps $f\colon U \to V$ and $g\colon V \to W$. The sequence

$$0 \longrightarrow U \xrightarrow{f} V \xrightarrow{g} W \longrightarrow 0$$

is a **short exact sequence** if:

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Think of U as a "subspace" of V and W as the "quotient space" V/U.

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Prototypical Example

Let $(U, V, W) = (\mathbb{R}^n, \mathbb{R}^{n+m}, \mathbb{R}^m)$.

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In particular

$$0 \longrightarrow \mathbb{R}^n \xrightarrow{\iota} \mathbb{R}^{n+m} \xrightarrow{\pi} \mathbb{R}^m \longrightarrow 0$$

is a short exact sequence.

Short Exact Sequences

Consider linear operators $a\colon U \to U$, $b\colon V \to V$, and $c\colon W \to W$ that intertwine with f and g (i.e. $f \circ a = b \circ f$ and $c \circ g = g \circ b$):

$$\begin{tikzcd} 0 \arrow[r] & U \arrow[r, "f"] \arrow[loop below, "a"] & V \arrow[r, "g"] \arrow[loop below, "b"] & W \arrow[r] \arrow[loop below, "c"] & 0 \end{tikzcd}$$

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Lemma

If maps a, b, and c intertwine with f and g, then

$$\operatorname{tr}(a) - \operatorname{tr}(b) + \operatorname{tr}(c) = 0.$$

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Recall that χ_ℓ^n is the trace of an operator (i.e. a character) on W_ℓ^n.

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Idea

If we can find a short exact sequence of $\mathrm{TL}_4(0)$-modules

$$0 \longrightarrow W_4^4 \longrightarrow W_2^4 \longrightarrow W_0^4 \longrightarrow 0,$$

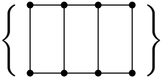
then the alternating sum for $n = 4$

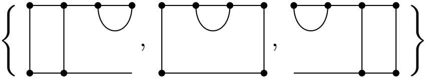
$$\chi_0^4(\hat{L}) - \chi_2^4(\hat{L}) + \chi_4^4(\hat{L}) = \sum_{\substack{\ell=0,\\ \ell \text{ even}}}^{4} (-1)^{\ell/2} \chi_\ell^4(\hat{L}) = 0$$

follows immediately!

When $n = 4$

Also recall that:

$$W_4^4 = \text{span}$$


$$W_2^4 = \text{span}$$


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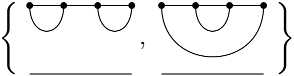

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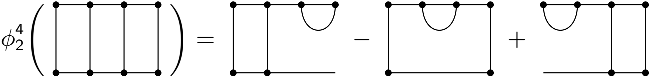

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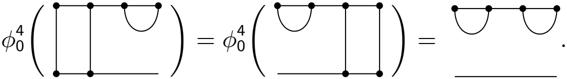

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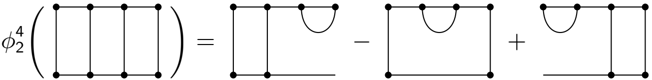

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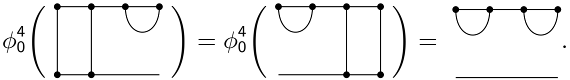

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This sheds light on the Overarching Question for $n = 4$.

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Let $V_1, V_2, \ldots, V_n$ be a collection of vector spaces with linear maps $f_i \colon V_{i-1} \to V_i$. Along with $V_0 = V_{n+1} = 0$, the sequence

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Stronger Lemma

If the V_i form an exact sequence and the linear operators $a_i \colon V_i \to V_i$ all intertwine, then

$$\operatorname{tr}(a_1) - \operatorname{tr}(a_2) + \operatorname{tr}(a_3) - \cdots \pm \operatorname{tr}(a_n) = \sum_{i=1}^{n} (-1)^{i-1} \operatorname{tr}(a_i) = 0.$$

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Main Result

Main Theorem (L.)

We have **explicit constructed** an exact sequence of standard modules of the Temperley-Lieb algebra $\mathrm{TL}_n(0)$ at zero:

$$0 \xrightarrow{\phi_n^n} W_n^n \xrightarrow{\phi_{n-2}^n} W_{n-2}^n \xrightarrow{\phi_{n-4}^n} \cdots \xrightarrow{\phi_2^n} W_2^n \xrightarrow{\phi_0^n} W_0^n \longrightarrow 0.$$

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This resolves the Overarching Question!

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- We also extend our appreciation to the MIT PRIMES-USA program and its organizers—Dr. Slava Gerovitch, Dr. Tanya Khovanova, and Prof. Pavel Etingof—for creating this research opportunity and providing us with resources and opportunities that would otherwise have been unimaginable.

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Q & A

THANK YOU!

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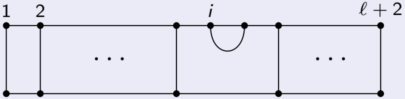
$\delta_i =$  .

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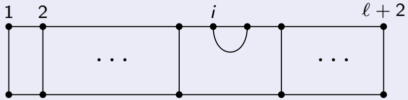

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The element $x\delta_i \in W_\ell^n$ is exactly the diagram formed by joining the ith and $(i+1)$th leftmost points on the bottom.

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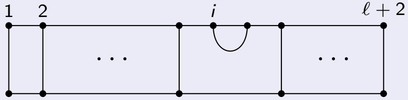

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Theorem (L.)

The maps $\phi^n_\ell : W^n_{\ell+2} \to W^n_\ell$ are homomorphisms that constitute a long exact sequence on the standard $\mathrm{TL}_n(0)$-modules given by

$$0 \xrightarrow{\phi^n_n} W^n_n \xrightarrow{\phi^n_{n-2}} W^n_{n-2} \xrightarrow{\phi^n_{n-4}} \cdots \xrightarrow{\phi^n_2} W^n_2 \xrightarrow{\phi^n_0} W^n_0 \longrightarrow 0.$$

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However, the above result only implies the existence of the L_ℓ^n, without detailing the structure of such modules.