Extremal Structural Results for Feedback Arc Sets and Graph Inversions

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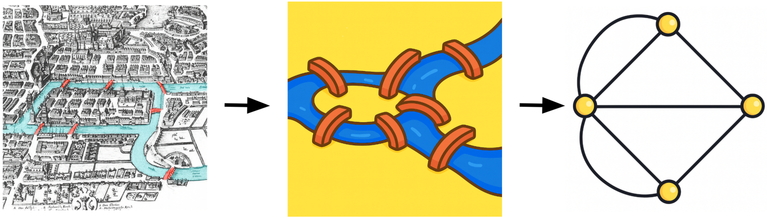
Introduction — Graphs

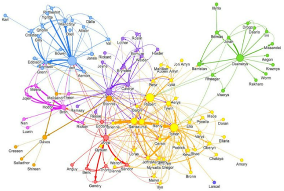
Graph: A collection of vertices and edges.



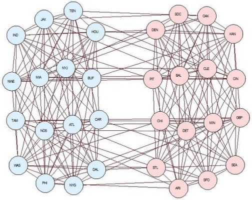
The "Seven Bridges of Königsberg" in graphs

Introduction — Graphs

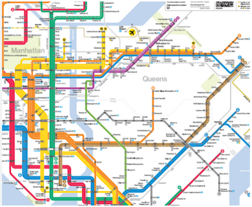
Graph applications:



Social network analysis



Game scheduling

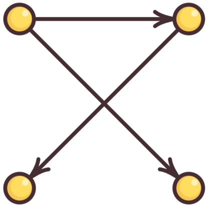


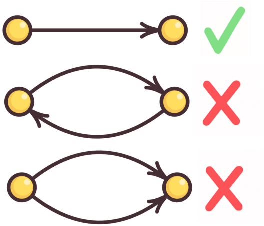
Subway planning

Introduction — Graphs

All graphs in this presentation are oriented simple graphs.

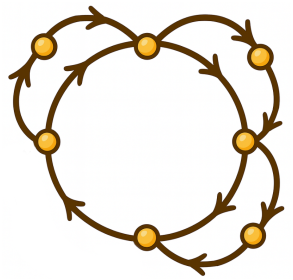
- Directed graph (digraph).
- At most one edge between two vertices.



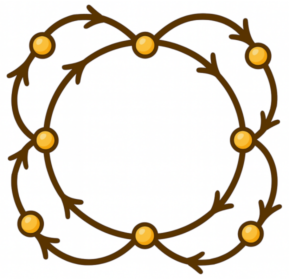


Introduction — Feedback Arc Set

Feedback arc set: A set of edges where the removal will break all cycles. The size of the minimum feedback arc set is $\beta(G)$.

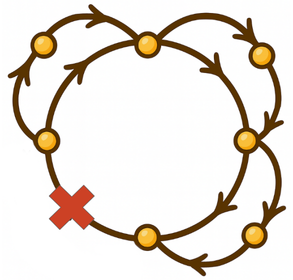


$\beta(G) = 1$

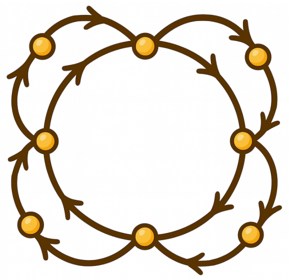


$\beta(G) = 2$

Feedback arc set: A set of edges where the removal will break all cycles. The size of the minimum feedback arc set is $\beta(G)$.



$\beta(G) = 1$



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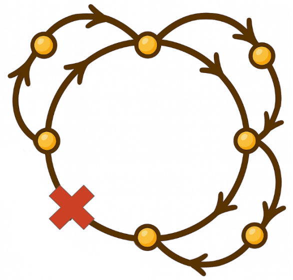

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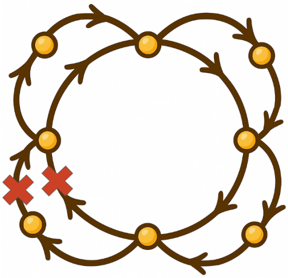


$\beta(G) = 2$

Motivation

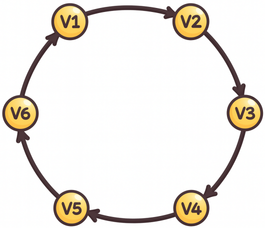
- Many systems must be acyclic.
- $\beta(G) =$ minimum cost to acyclic.
- Finding $\beta(G)$ is NP-hard!
  - Research interests on
    - Upper/lower bounds with cheaper costs.
    - Extremal structures under various conditions.
- Our project focuses on edge removal and inversion to break all cycles.

Minimum Feedback Arc Set Size

Theorem (Folklore)

For a digraph G with m edges, the minimum feedback arc set size $\beta(G) \leq \frac{m}{2}$.

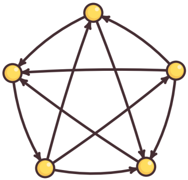
- $v_i \to v_j$ is a *forward edge* if $i < j$, and it is a *backward edge* otherwise.
- Remove all the forward edges or backward edges.



$$V_1 \to V_2 \to V_3 \to V_4 \to V_5 \to V_6 \to V_1$$

Extremal Results — with Tournament

Tournament: A digraph with exactly one edge between every two vertices.



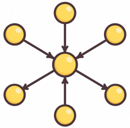
If the graph is a tournament:

Theorem (Poljak, Rödl, and Spencer '88)

For a tournament T, and some constant $c > 0$, the minimum feedback arc set size $\beta(T) \leq \frac{m}{2} - cm^{\frac{3}{4}}$.

Extremal Results — with Maximum Degree

Maximum degree: The maximum number of edges connected to any single vertex in a graph.



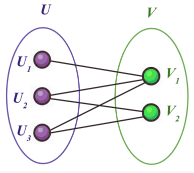
Maximum degree $\Delta = 6$.

Theorem (Berger and Shor '90)

For all digraphs G with m edges and Δ being the maximum degree, the minimum feedback arc set size $\beta(G) \leq \frac{m}{2} - c\frac{m}{\sqrt{\Delta}}$.

Extremal Results — with Structure Restriction

Bipartite graph: A graph with vertices split into two groups such that every edge connects one vertex from each group.



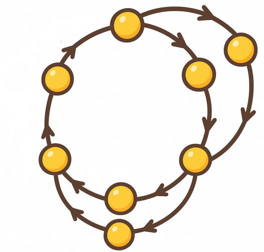
Theorem (Fox, Himwich, and Mani '22)

For each bipartite digraph B, there are $c, \epsilon > 0$ depending on B such that the following holds. If the digraph G has m edges and is B-free, then

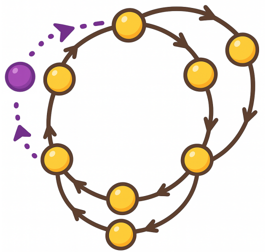
$$\beta(G) \leq \frac{m}{2} - cm^{3/4+\epsilon}.$$

Extremal Results — with Shortest Cycle Size

A *r*-free graph is a graph where there is no cycle of size r or less.



$n = 8$
$r = 5$
$\beta(G) = 1$



$n = 9$
$r = 5$
$\beta(G) = 2$

Extremal Results — with Shortest Cycle Size

$\gamma(G)$: The number of non-adjacent unordered pairs of vertices of G.

Conjecture (Sullivan '08)

All r-free digraphs satisfy $\beta(G) \leq \frac{2\gamma(G)}{(r+1)(r-2)}$.

Extremal Results — with Shortest Cycle Size

Theorem (Fox, Himwich, and Mani '22)

There exists a family of r-free digraphs G with n vertices where $\beta(G) \geq \frac{n^2}{(r+1)^2}$.

Conjecture (Fox, Himwich, and Mani '22)

If a digraph G with n vertices is $(r-1)$-free and $r > \frac{2n}{3}$, then $\beta(G) \leq 1$.

Result

Theorem

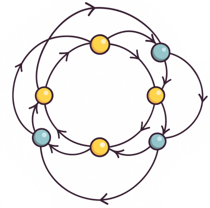
If a digraph G with n vertices is $(r-1)$-free and $r > \frac{2n}{3}$, then $\beta(G) \leq 1$.

Result

Bypass vertex: For cycle C and path $P = (c_1, c_2) \in C$, if there exists a path $c_1 \to v \to c_2$, then $v \notin C$ is a bypass vertex of P.
X: A subgraph with two paths $P_1, P_2 \in C$ sharing a bypass vertex, and $V(P_1) \cap V(P_2) = \emptyset$.

Theorem

Any X-free, $(r-1)$-free digraph G with $n \leq 2r - 2$ satisfies $\beta(G) \leq 2$.



A graph with $n = 7$ and $r = 4$, where $n = 2r - 1$, we find $\beta(G) = 3$.

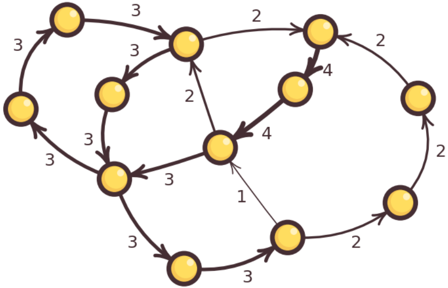
Finding Minimum Feedback Arc Set

- Finding the minimum feedback arc set is NP-hard.
- Finding the minimum feedback arc set while $\beta(G) = 1$.
  - Remove one edge at a time - $O(E)$.
  - Check for cycles with depth first search - $O(|V| + |E|)$.
  - Overall - $O(|E| \cdot (|V| + |E|))$.

Result

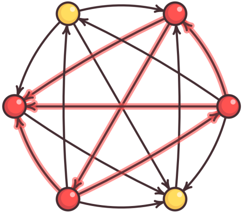
Algorithm

Presented a linear-time algorithm, running in $O(|V| + |E|)$, to identify the minimum feedback arc set while $\beta(G) = 1$.



Inversion Number

Invert k vertices: Reverse all edges between any two of those k vertices.



Inversion number: The minimum number of times we need to invert at most k vertices to remove all cycles in a tournament T, denoted as $\mathrm{inv}_k(T)$.

Inversion Number Bound

If inverting edges other than removing:

Theorem (Yuster '25)

For a digraph with n vertices, the function $\mathrm{inv}_3(n)$ satisfies $\mathrm{inv}_3(n) < \frac{257}{2592}(1+o(1))n^2$.

Theorem (Yuster '25)

For a digraph with n vertices, the function $\mathrm{inv}_k(n)$ satisfies $(1+o(1))\,\mathrm{inv}_k(n)/n^2 \in [\frac{1}{2k(k-1)} + \delta_k, \frac{1}{2\lfloor \frac{k^2}{2} \rfloor} - \epsilon_k]$.

Result

Theorem

The inversion number of a tournament T with n vertices has an upper bound of $\text{inv}_4(n) \leq \frac{11n^2}{180} + o(n^2)$.

Improved from $\text{inv}_4(n) \leq \frac{n^2}{16} + o(n^2)$.

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