Diameter Bounds for Friends-and-strangers Graphs

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Overview

Overview

Motivation

The 15 puzzle involves sliding tiles around in a grid with one empty tile until the tiles reach a desired configuration.

Example

1	2	3	4
5	6	7	8
9	10	11	12
13	14	15	

$\rightarrow$

1	2	3	4
5	6	7	8
9	10	11	
13	14	15	12

$\rightarrow$

1	2	3	4
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Two questions come to mind:

1. Given a starting configuration, which other configurations can we reach?
2. What is the maximum number of moves we need to reach another configuration?

Motivation

Two questions come to mind:

1. Given a starting configuration, which other configurations can we reach? Well-understood.
2. What is the maximum number of moves we need to reach another configuration? The main topic of this talk.

We can ask these questions about sliding puzzles over any movement graph.

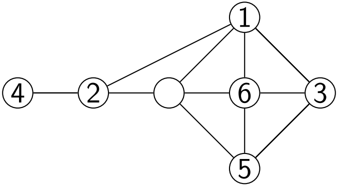


Figure: A sliding puzzle on 7 vertices.

We'll use Friends-and-Strangers graphs to study sliding puzzles and their extensions.

Overview

Graphs

Definition

A graph G is a set of vertices (denoted by $V(G)$) connected by a set of edges (denoted by $E(G)$).

Example

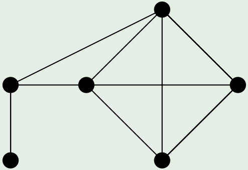


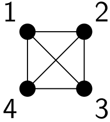
Figure: A graph G on 6 vertices.

Special Classes of Graphs

By default, our graphs on n vertices have vertex set $[n] = \{1, 2, \ldots, n\}$.

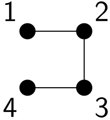
- The graph K_n has edge set $\{(i, j) \mid i \neq j \text{ and } i, j \in [n]\}$.

K_4:



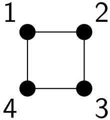
- The graph Path_n has edge set $\{(i, i+1) \mid i \in [n-1]\}$.

Path_4:



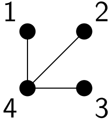
- The graph Cycle_n has edge set $\{(i, (i+1) \bmod n) \mid i \in [n]\}$.

Cycle_4:



- The graph Star_n has edge set $\{(i, n) \mid i \in [n-1]\}$.

Star_4:



Connectivity

Definition

A graph is connected if there exists a path between any two of its vertices. Otherwise, it is disconnected.

Definition

A maximal connected subgraph is called a connected component.

Example

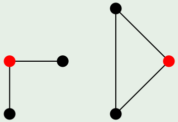


Figure: The above graph is disconnected, because there is no path between the two red vertices.

Connectivity

Definition

A graph is connected if there exists a path between any two of its vertices. Otherwise, it is disconnected.

Definition

A maximal connected subgraph is called a connected component.

Example

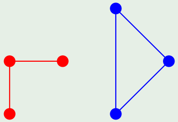


Figure: The two connected components of the above graph are colored in red and blue.

Diameter of a graph

Definition

The distance between two vertices u and v of a graph G is equal to the minimum number of edges needed in a path from u to v.

Definition

The diameter of a graph G is the maximum possible distance between vertices u and v over all pairs of vertices (u, v) in G.

Example

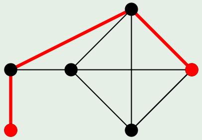


Figure: The diameter of the graph G is 3.

Other important definitions

Definition

The degree of a vertex v in a graph G is the number of edges with an endpoint at v. The minimum degree of a graph G, denoted as $\delta(G)$, is the smallest degree over all vertices in G.

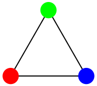
Definition

An Erdős-Rényi random graph is a graph on n vertices for which each edge exists with independent probability p. We'll denote the probability distribution of choosing this graph as $G(n, p)$.

Overview

Friendly Swaps

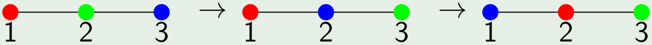
Let X and Y be the graphs shown below:

X: 

Y: 

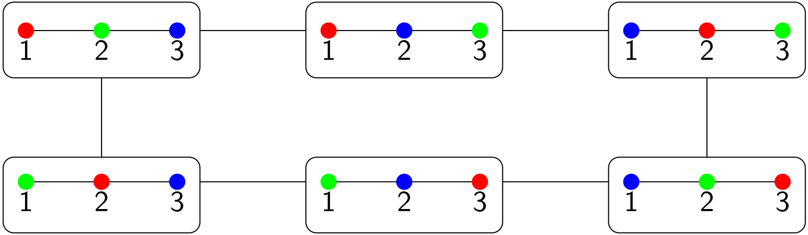
- We think of X as a friendship graph and Y as a movement graph.
- Place the people in X onto the vertices of Y such that one person occupies each position. Then, we can swap two friends in a friendly swap if they are standing on adjacent positions.

Example



Friends-and-strangers Graphs

In the following image, we visualize all six possible configurations of the people in X on the graph Y:

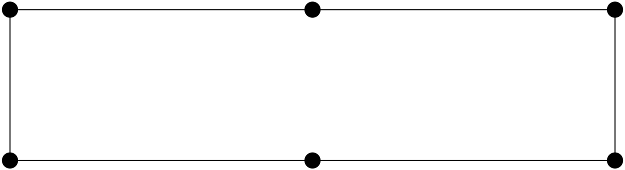


We add an edge between two configurations if it is possible to reach one from the other through a friendly swap.

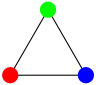
Friends-and-strangers Graphs

In the following image, we visualize all six possible configurations of the people in X on the graph Y:

$\mathrm{FS}(X, Y)$:



X:



Y:

We add an edge between two configurations if it is possible to reach one from the other through a friendly swap.

We'll call this the friends-and-strangers graph of X and Y, denoted as $\mathrm{FS}(X, Y)$. Here, we've shown that $\mathrm{FS}(\mathrm{Cycle}_3, \mathrm{Path}_3) = \mathrm{Cycle}_6$.

The 15 puzzle, revisited

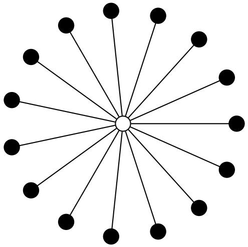
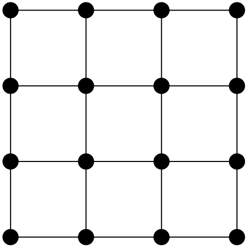
Recall the 15 puzzle from before:

1	2	3	4
5	6	7	8
9	10	11	12
13	14	15	

1. We can think of each of the tiles (including the blank space) as people and our grid as the possible positions for our people.
2. Then, the blank space is friends with everyone.

The 15 puzzle, revisited

Therefore, we can think of our friendship graph and our position graph as the graphs below:

$X = \mathrm{Star}_{16}$:  $Y = \mathrm{Grid}_{4\times 4}$: 

Hence, the graph $\mathrm{FS}(\mathrm{Star}_{16}, \mathrm{Grid}_{4\times 4})$ can be used to model the different configurations of the 15 puzzle.

More generally, if Y is a graph with n vertices, then $\mathrm{FS}(\mathrm{Star}_n, Y)$ models the different configurations of the sliding puzzle on Y.

Overview

Prior Results on Connectivity

The connectivity of Friends-and-Strangers graphs is their most extensively-studied property.

- Defant–Kravitz (2021) introduces Friends-and-Strangers graphs and shows several foundational results.
- Wilson (1974) characterizes the connectivity of $\mathrm{FS}(\mathrm{Star}_n, Y)$.
- Bangachev (2022) proves certain bounds on $\delta(X)$ and $\delta(Y)$ for which $\mathrm{FS}(X, Y)$ must be connected.
- Wang–Chen (2023) proves bounds on p and q for which $\mathrm{FS}(G(n,p), G(n,q))$ is connected with high probability.

Overview

Jeong's Results on Diameters

Fact

The diameter of $\mathrm{FS}(X, Y)$ is equal to the maximum number of friendly swaps needed to move from one configuration to another.

Jeong shows several results on the diameter of friends-and-strangers graphs.

Theorem (Jeong, 2022)

Let X and Y be graphs on n vertices. Then, the diameter of any connected component of $\mathrm{FS}(X, Y)$ is bounded from above by:

- $\binom{n}{2}$ *when* $X = \mathrm{Path}_n$.
- $8n^4(1 + o(1))$ *when* $X = \mathrm{Cycle}_n$.
- $2n^2 - 5n + 3$ *when* $X = K_n$.

Jeong's main result

Theorem (Jeong, 2022)

There exist families of graphs X_n and Y_n for which there is a connected component in $\mathrm{FS}(X_n, Y_n)$ with diameter $e^{\Theta(n)}$.

- Observe that if X and Y have n vertices, then $\mathrm{FS}(X, Y)$ has $n! = e^{O(n \log n)}$ vertices.
- Therefore, exponential diameter could be possible, and is shown to be possible by Jeong's theorem.
- No nontrivial upper bound is known. Superexponential diameter could be possible too.

Overview

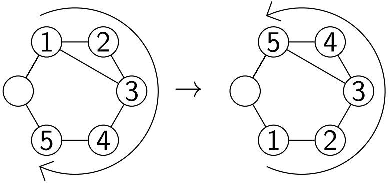
Star Graphs

Theorem (A.–Li, 2025+)

Let Y be a graph on n vertices. Then, the diameter of any connected component of $\mathrm{FS}(\mathrm{Star}_n, Y)$ is $O(n^4)$.

This implies that if a sliding puzzle on n vertices is solvable, then it is solvable in $O(n^4)$ moves.

We have also constructed a lower bound:



takes $\Omega(n^3)$ friendly swaps to complete.

Star Graphs

Theorem (A.–Li, 2025+)

Let Y be a graph on n vertices. Then, the diameter of any connected component of $\mathrm{FS}(\mathrm{Star}_n, Y)$ is $O(n^4)$.

Key step: we make moves in θ-subgraphs.

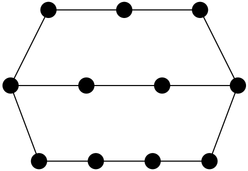


Figure: θ-graphs are literally shaped like θ!

Complete Graphs

Recall that Jeong proved the following theorem:

Theorem (Jeong, 2022)

Let Y be a graph on n vertices. Then, the diameter of any connected component of $\mathrm{FS}(K_n, Y)$ is at most $2n^2 - 5n + 3$.

We improve this to the following bound:

Theorem (A.–Li, 2025+)

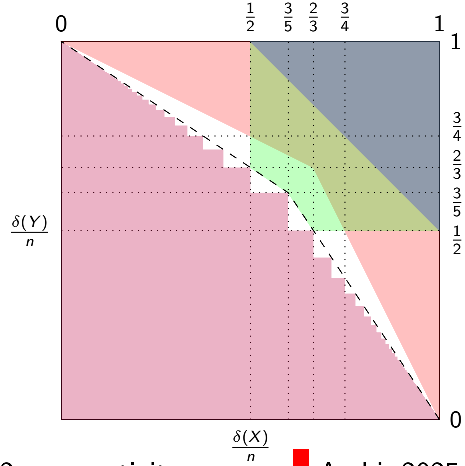
Let Y be a graph on n vertices. Then, the diameter of any connected component of $\mathrm{FS}(K_n, Y)$ is at most $\binom{n}{2}$.

This bound is tight:

Theorem (A.–Li, 2025+)

The diameter of $\mathrm{FS}(K_n, \mathrm{Path}_n)$ is $\binom{n}{2}$.

Diameter Bounds Based on Minimum Degree



Bangachev, 2022: connectivity.

Bangachev, 2022: connectivity.

Bangachev, 2022: disconnection.

A.–Li, 2025: $O(n^6)$ diameter.

A.–Li, 2025: $O(n^2)$ diameter.

Conjectured Threshold.

Random Graphs

Theorem (A.–Li, 2025+)

Let p and q be probabilities for which $pq \geq \frac{100 \log n}{n}$. Let X and Y be chosen from $G(n, p)$ and $G(n, q)$, respectively. Additionally, let σ and τ be any two configurations of the people in X on the vertices in Y. Then, with probability $1 - o(n^{-2})$, the distance between σ and τ in $\mathrm{FS}(X, Y)$ is at most $O(n^6)$.

Note that this does not imply that $\mathrm{FS}(X, Y)$ has low diameter.

Conjecture

Let ε be a positive constant, let p and q be probabilities such that $pq \geq n^{\varepsilon}/n$, and let X and Y be graphs chosen from $G(n, p)$ and $G(n, q)$, respectively. Then, $\mathrm{FS}(X, Y)$ is connected and has $\mathrm{poly}(n)$ diameter with high probability.

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