Pattern Avoidance in Sequences

Shihan Kanungo
Mentor: Prof. Jesse Geneson (SJSU)

SJSU

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Outline

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In the field of extremal combinatorics, we ask questions like

Extremal Question

What is the largest possible size of an object which avoids a given forbidden substructure?

For example:

- What is the largest possible graph on n vertices that does not contain K_3 as a subgraph? (Turan's Theorem & the forbidden subgraph problem)
- What is the largest possible group of people, such that for any set of k people, they are not all friends or not all strangers? (Ramsey Theory)
- What is the largest possible subset of $\{1, \ldots, n\}$ that does not contain a k-term arithmetic progression? (Szemeredi's Theorem)

Saturation

The saturation question is a bit more complicated.

Saturation Question

What is the SMALLEST possible structure that avoids a given forbidden substructure, BUT making it larger in any way induces a copy of the forbidden structure?

In other words: the *minimum* size of a *maximal* structure, rather than the *maximum* size.

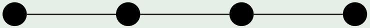
Saturation for Graphs

Definition

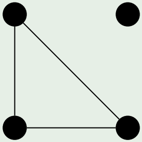
Let G and H be graphs. We say G is H-saturated if G avoids H as a subgraph, but adding any new edge to G induces a copy of H.

Example

Consider

$H =$ 

The graph



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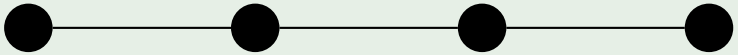
$H =$ 

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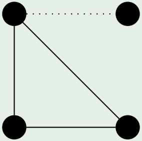

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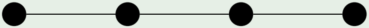
$H =$ 

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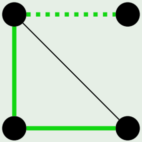

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Dichotomy

Definition

The saturation function $\mathrm{Sat}(n, H)$ is the minimum number of edges in a H-saturated graph on n vertices.

$\mathrm{Sat}(n, H)$ exhibits an dichotomy:

Theorem (Kászonyi-Tuza, 1986)

We have $\mathrm{Sat}(n, H) = O(1)$ *or* $\mathrm{Sat}(n, H) = \Theta(n)$*.*

In many other settings, it has been seen that the saturation function exhibits the same dichotomy.

Outline

Sequence Saturation

We use "letters" to refer to the terms in a sequence.

Definition

Let s, u be two sequences. We say that s contains a copy of u if s has a subsequence that can be turned into u by a *one-to-one* renaming of letters.

Example

The sequence $s = \mathbf{1}, \mathbf{2}, 3, 2, 3, \mathbf{1}, \mathbf{2}$ contains a copy of $u = abab$:

$$\begin{array}{ccccccc} 1 & 2 & 3 & 2 & 3 & 1 & 2 \\ a & b & & & & a & b \end{array}$$

However, $s = 1, 2, 3, 2, 1$ does not.

Definition

A sequence s is r-sparse if every consecutive r letters are pairwise distinct.

Example

The sequence $s = 1, 2, \mathbf{3}, \mathbf{2}, \mathbf{3}, 1, 2$ is 2-sparse, but not 3-sparse.

Definition

Let u be a sequence with r distinct letters. A sequence s is u-saturated if s avoids u, s is r-sparse, and inserting any new letter into s either induces a copy of u or violates r-sparsity.

If we dropped the r-sparsity condition, we would have arbitrarily long sequences like $1, 1, \cdots$ which avoid u.

Example

If $u = abca$ then the sequence $s = 1, 2, 3$ is u-saturated.

First, $r = 3$ since u has 3 distinct letters. Now we check:

s **avoids** u**:** Evident.

s **is 3-sparse:** Evident.

Saturation: Suppose we insert a 1 into s. The possibilities are:

$$\color{green}{\mathbf{1}}\color{black}\quad 1 \quad 2 \quad 3 \quad \text{(violates 3-sparsity)}$$

$$1 \quad \color{green}{\mathbf{1}}\color{black}\quad 2 \quad 3 \quad \text{(violates 3-sparsity)}$$

$$1 \quad 2 \quad \color{green}{\mathbf{1}}\color{black}\quad 3 \quad \text{(violates 3-sparsity)}$$

$$\begin{matrix} 1 & 2 & 3 & \color{green}{\mathbf{1}} \\ a & b & c & a \end{matrix} \quad \text{(has a copy of } u\text{)}$$

Similar checks for the other letters.

Definition

The saturation function $\mathrm{Sat}(n, u)$ is the length of the shortest u-saturated sequence with n distinct letters.

In 2021, Anand, Geneson, Kaustav, and Tsai conjectured

Conjecture

We have $\mathrm{Sat}(n, u) = O(n)$.

This implies the dichotomy $\mathrm{Sat}(n, u) = O(1)$ or $\Theta(n)$. They proved

Theorem (Anand-Geneson-Kaustav-Tsai, 2021)

We have $\mathrm{Sat}(n, u) = O(n)$ *for all sequences* u *with two distinct letters.*

However, the cases for u having ≥ 3 distinct letters remained completely open.

Outline

Algorithm

Consider the following algorithm:

Input: Alphabet $A = \{1, \ldots, n\}$, forbidden sequence u
Output: u-saturated sequence
Initialize the sequence: $s \leftarrow 1, 2, \ldots, r-1$ $\triangleright$ Initial sequence avoids u
while it is possible to extend the sequence **do**
  for each letter $x \in A$ **do**
    if x can be *properly inserted* into s **then**
      Insert x appropriately into s to form s' $\triangleright$ Smallest x, leftmost position
      Update $s \leftarrow s'$ $\triangleright$ New sequence
      break $\triangleright$ Exit loop after the first valid insertion
    end if
  end for
end while
Return s $\triangleright$ Final sequence is u-saturated

Algorithm

The output of the algorithm:

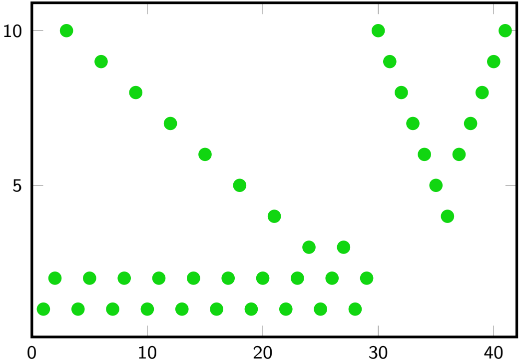

Figure: Algorithm on $u = abcacbc$.

Here, we represent $s = s_1 \cdots s_\ell$ by plotting the points (i, s_i). Using this pattern, we get $\mathrm{Sat}(n, abcacbc) = O(n)$!

Some more pictures:

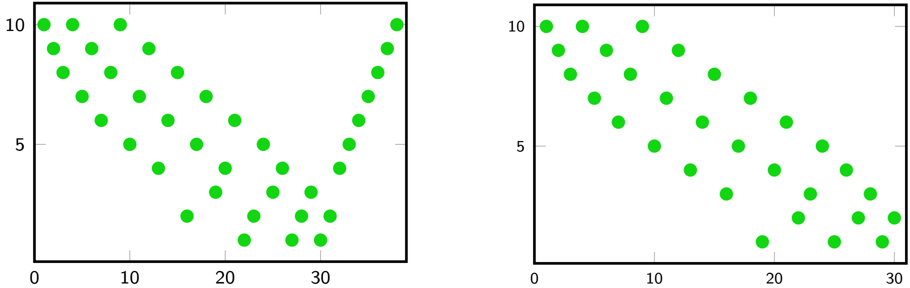

Figure: Algorithm on $u = abbacac$ (left) and $abcacba$ (right).

And even more:

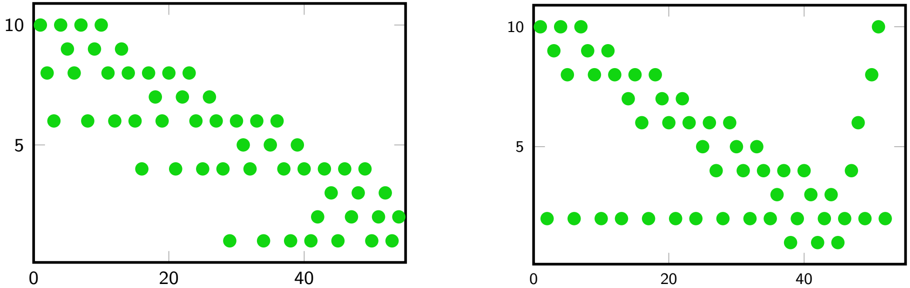

Figure: Algorithm on $u = abcacbacb$ (left) and $abcbacbac$ (right).

This lets us resolve the conjecture for many specific sequences u.

Say u is irreducible if u cannot be decomposed into sequences $u = u_1 u_2$ such that u_1 and u_2 have no letters in common.

Theorem (Kanungo, 2025+)

If u is irreducible and of the form $aa \cdots bb$, then $\mathrm{Sat}(n, u) = O(n)$.

Suppose u is a sequence on 3 letters, and $u = abc \cdots xyz$ where a, b, c are distinct. Define

$$f_0(u) = \#\{\,\text{consecutive pairs of the form } ab, bc, ca\,\},$$
$$f_1(u) = \#\{\,\text{consecutive pairs of the form } ac, ba, cb\,\}.$$

Theorem (Kanungo, 2025+)

Let $u = abc \ldots xyz$ be a three-letter sequence with a, b, c distinct. Suppose

$$xyz \in \{abc, bca, cab\}, \qquad f_0(u) \geq f_1(u) + 5.$$

Then $\mathrm{Sat}(n, u) = O(n)$.

Corollary

For any sequence u on 3 letters, $\mathrm{Sat}(n, (abc)u(abc)^t) = O(n)$ *for large enough t.*

Acknowledgments

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- The PRIMES organizers, for making this experience possible.
- My parents, without whom none of this would be possible.

Questions?

Thank You!