Applications of Graphical Representations of Temperley-Lieb Types

Feodor Yevtushenko and Son Nguyen (MIT)

MIT PRIMES Conference, October 18, 2025

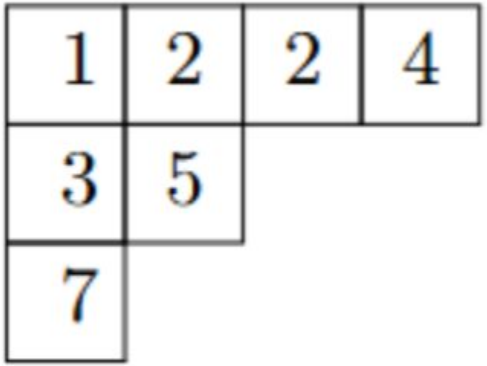
Definitions

Def: Semi-Standard Young Tableaux (SSYT)

A semi-standard Young Tableaux is defined as a grid of numbers with the following properties:

- Rows are weakly increasing rightward, while columns are strictly increasing downward.
- All rows are left-aligned, and all columns are top-aligned.

We define the **shape** λ of the SSYT to list the number of cells in each row (here λ=(4,2,1)) and the **content** c to list the number of each element present in the SSYT. In other words, c_i is the number of elements with value i. Here, c=(1,2,1,1,1,0,1).



Def: Schur Functions

For a given SSYT shape λ, we define the *Schur function* s_λ by summing the over the contents c of all SSYTs T of shape λ:

$$s_\lambda = \sum_{\text{SSYT } T \text{ of shape } \lambda} \omega(T).$$

where we define:

$$\omega(T) = x_1^{c_1} x_2^{c_2} x_3^{c_3} \ldots$$

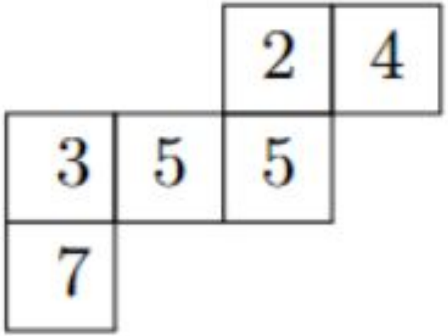
Again, c_i represent the content of T. This turns out to be an infinite-variable **symmetric** polynomial in the c_i.

1	2	2	4
3	5		
7			

Def: *Skew* Semi-Standard Young Tableaux (SSYT)

- Same as the “normal” SSYT, except the shape has a smaller SSYT removed.
- The shape is now λ/μ, where the skew SSYT has entries in columns $(\mu_i, \lambda_i]$ in row i.
  - λ represents the right boundary
  - μ represents the left boundary
- We can analogously define the skew-Schur function $s_{\lambda/\mu}$ by summing over the contents of every valid SSYT:

$$s_{\lambda/\mu} = \sum_{\text{skew SSYT } T \text{ of shape } \lambda/\mu} \omega(T)$$



Littlewood-Richardson Coefficients

We have the following identity: $$s_{\lambda/\mu} = \sum_{\nu} c^{\lambda}_{\mu,\nu} s_{\nu}$$

Here, the $c^{\lambda}_{\mu,\nu}$ are the Littlewood-Richardson coefficients, which count the number of skew-Schur tableaux with shape λ/μ and content ν (and are thus all nonnegative). A corollary of this is that Schur functions form a basis for skew-Schur functions.

We call a group of symmetric functions *Schur positive* if they can be represented as a combination of Schur functions with a nonnegative coefficient on each Schur function.

More on this soon!

The Jacobi-Trudi Identity

The Jacobi-Trudi identity states that $s_{\lambda/\mu} = \det\left(h_{\lambda_i-\mu_j-i+j}\right)_{1\leq i,j\leq n}$, where we have:

$h_n = \sum_{0<i_1\leq i_2\leq i_3\leq\cdots\leq i_n} x_{i_1}x_{i_2}x_{i_3}\ldots x_{i_n}$ and $h_\lambda = h_{\lambda_1}h_{\lambda_2}h_{\lambda_3}\ldots.$

This gives us an explicit algebraic form for the skew-Schur functions. We can prove this by bijecting the determinant to groups of paths connecting the sets $(n-i+\mu_i,\infty)$ and $(n-i+\lambda_i,1)$, with the weight ω of a path π_i being the product of x_b over all steps $(a,b)\rightarrow(a+1,b)$. Specifically, by the Lindstrom-Gessel-Viennot lemma, it suffices to sum over all noncrossing paths, giving us the desired expansion.

Temperley-Lieb Immanants

In the wiring from the previous slide, modify the left and right endpoints L_i and R_i to be $(\mu_i, 1)$ and (λ_i, ∞) respectively, and relax the noncrossing condition to allow at most two paths to intersect per vertex. As before, let $\omega(H) = \prod_{i=1}^{n} \omega(\pi_i)$ be the weight of the wiring H.

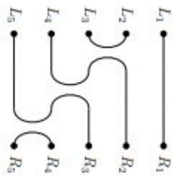
Next, at all intersections, disconnect all the paths and connect both incoming paths to each other, and also connect both outgoing paths to each other. Let $\epsilon(H)$ be the number of loops in the graph, and let τ be the *type* ("connectivity") of the resulting graph - in other words, which L_i and R_i connect to each other.

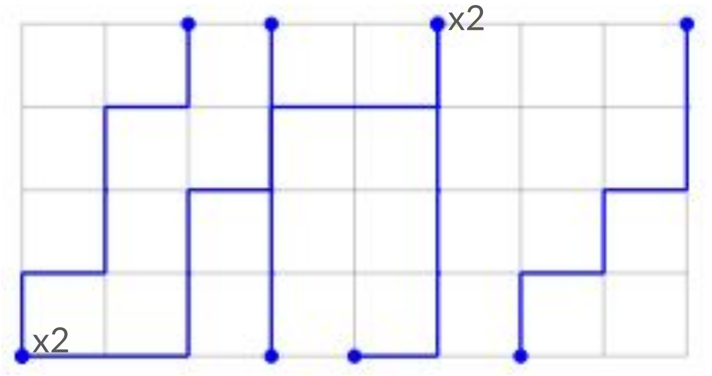
We define the Temperley-Lieb immanant as the following sum over all configurations with the same type and path matrix:

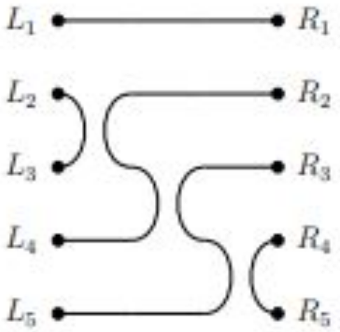
$$\mathrm{Imm}_{\tau}^{\mathrm{TL}}(A) = \sum_{H} 2^{\epsilon(H)} \omega(H)$$

Wiring Example

Here is an example of a wiring and its corresponding Temperley-Lieb type, for sequences (9,6,6,4,3) and (7,5,4,1,1):

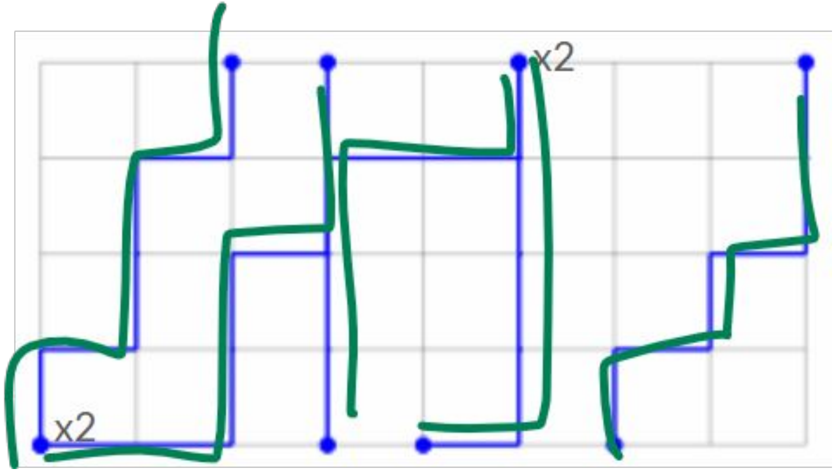






Wiring Example

Here is an example of a wiring and its corresponding Temperley-Lieb type, for sequences (9,6,6,4,3) and (7,5,4,1,1):



Jacobi-Trudi Matrices

We can also recast the skew-Schur functions explicitly in terms of symmetric functions:

$$s_{\lambda/\mu} = \det\left(h_{\lambda_i-\mu_j-i+j}\right)_{1\leq i,j\leq n}$$

This is known as the Jacobi-Trudi identity and is central to motivating our main structure, shuffle tableaux.)

(Here, $h_n = \sum_{0<i_1\leq i_2\leq i_3\leq\cdots\leq i_n} x_{i_1}x_{i_2}x_{i_3}\ldots x_{i_n}$ and $h_\lambda = h_{\lambda_1}h_{\lambda_2}h_{\lambda_3}\ldots.$)

Jacobi-Trudi Matrices: Example

Here, we consider $\lambda = (7, 5, 4, 4), \quad \mu = (3, 3, 2, 1)$:

$$s_{\lambda/\mu} = \det \begin{pmatrix} h_4 & h_5 & h_7 & h_9 \\ h_1 & h_2 & h_4 & h_6 \\ 0 & 1 & h_2 & h_4 \\ 0 & 0 & h_1 & h_3 \end{pmatrix}$$

Jacobi-Trudi Matrices: Example

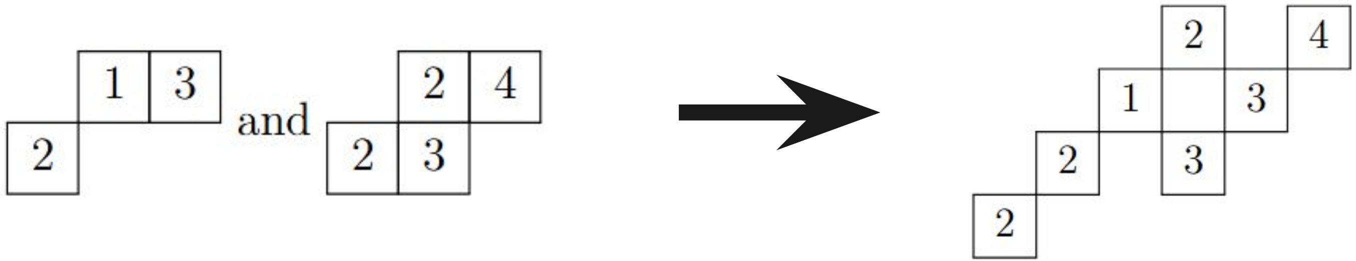
Here, we consider $\lambda = (7, 5, 4, 4), \quad \mu = (3, 3, 2, 1)$, but now look at minors:

$$s_{\lambda/\mu} = \det \begin{pmatrix} h_4 & h_5 & h_7 & h_9 \\ h_1 & h_2 & h_4 & h_6 \\ 0 & 1 & h_2 & h_4 \\ 0 & 0 & h_1 & h_3 \end{pmatrix}$$

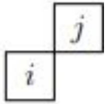
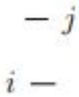
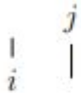
These minors are *themselves* determinants that correspond to skew-Schur functions and thus skew semi-standard Young tableaux. This motivates “breaking down” determinant expansions into two skew SSYTs.

Shuffle Tableaux

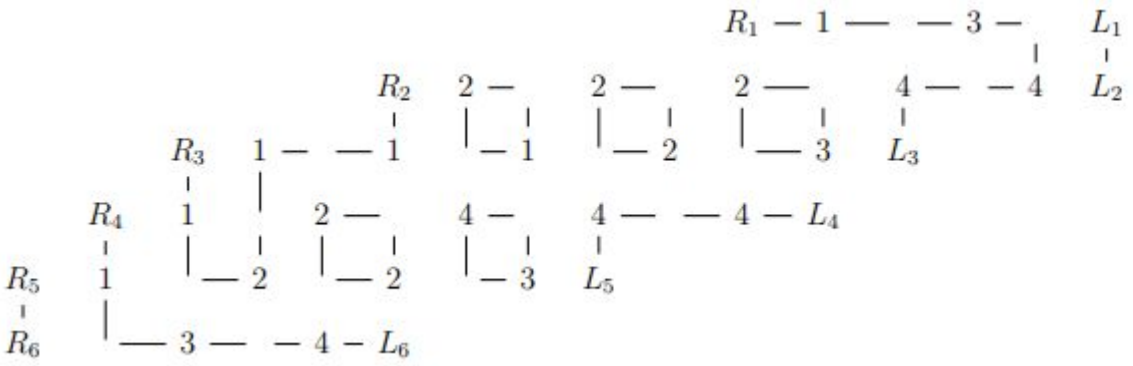
We can *interleave* two skew SSYTs, provided that they follow some basic rules:



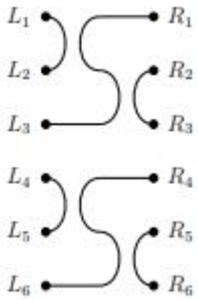
Temperley-Lieb Types

For each pair of cells , draw the lines  if i > j and  otherwise.

Draw out the lines for the entire grid and look at how the leftmost/rightmost elements in each row are connected. This left-right connectivity is the *Temperley-Lieb type* τ of the shuffle tableaux:



(left/right flipped by convention)



Yamanouchi Tableaux

By examining the *reading word* of the shuffle tableaux, we can pair values of *i* and *i+1* in the skew SSYT and change unmatched *i+1* values to *i*. One specific operation that does this is known as the E_i crystal operator. Skew SSYTs that do not change upon the application of any E_i are known as Yamanouchi tableaux. We have the following theorem, by Son and Pylyavskyy, which is the premise of our entire project:

> **Theorem 6.2.** *For any partitions μ, ν, any Temperley-Lieb type τ, and any partition λ, the coefficient of the Schur function s_λ in $\mathrm{Imm}_\tau^{\mathrm{TL}}(A_{\mu,\nu})$ is the number of Yamanouchi shuffle tableaux of shape $\mu \oslash \nu$, Temperley-Lieb type τ, and content λ.*

This generalization of Littlewood-Richardson coefficients allows us to investigate the Temperley-Lieb immanant through a purely elementary combinatorial perspective.

Temperley-Lieb Immanants

Temperley-Lieb immanants are a generalization of determinants that are central to many areas of enumerative combinatorics. One can calculate a given Temperley-Lieb immanant by summing over all shuffle tableaux with a given shape $\mu \oslash \nu$ and a given Temperley-Lieb type **τ**. In particular:

For any basis element τ of $TL_n(2)$ and partitions μ, ν, we have

$$\mathrm{Imm}_\tau^{\mathrm{TL}}(A_{\mu,\nu}) = \sum_{\substack{T \text{ of shape } \mu \oslash \nu \\ \psi(T)=\tau}} \omega(T).$$

- We include this definition here for completeness and for motivating the more direct representation of the Temperley-Lieb immanant on the next slide.
- For now, just focus on the *algebraic form* of the above definition.

Temperley-Lieb Immanants

Temperley-Lieb immanants are a generalization of determinants that are central to many areas of enumerative combinatorics. One can calculate a given Temperley-Lieb immanant by summing over all shuffle tableaux with a given shape $\mu \oslash \nu$ and a given Temperley-Lieb type **τ**. In particular:

For any basis element τ *of* $TL_n(2)$ *and partitions* μ, ν*, we have*

$$\mathrm{Imm}_\tau^{\mathrm{TL}}(A_{\mu,\nu}) = \sum_{\substack{T \text{ of shape } \mu \oslash \nu \\ \psi(T)=\tau}} \omega(T).$$

- Here, T can be thought of as a tableau and omega(T) is the “weight” (a geometrically motivated infinite dimensional polynomial) of the *wirings* that represents this tableaux.
- We include this definition here for completeness and for motivating the more direct representation of the Temperley-Lieb immanant on the next slide.
- For now, just focus on the *algebraic form* of the above definition.

Yamanouchi Tableaux

- If we force our shuffle tableaux to satisfy an additional minimality condition known as being *Yamanouchi*, we have the following theorem, by Nguyen and Pylyavskyy, which is the premise of our entire project:

Theorem 6.2. *For any partitions μ, ν, any Temperley-Lieb type τ, and any partition λ, the coefficient of the Schur function s_λ in $\mathrm{Imm}_\tau^{\mathrm{TL}}(A_{\mu,\nu})$ is the number of Yamanouchi shuffle tableaux of shape $\mu \oslash \nu$, Temperley-Lieb type τ, and content λ.*

- Notice that by restricting ourselves to Yamanouchi tableaux, the wiring weights reduce to Schur functions!
- This generalization of Littlewood-Richardson coefficients allows us to investigate the Temperley-Lieb immanant through a purely elementary combinatorial perspective.
- In particular, by examining what shuffle tableaux exist, we can infer what Schur functions have a nonzero coefficient in the Temperley-LIeb immanant!

Our *Initial* Main Conjecture

Fix a shape λ/μ for a shuffle tableaux and a Temperley-Lieb type τ. Consider all possible contents ν of such Yamanouchi shuffle tableaux, which form a set P. Once again, ν_i is the number of elements equal to i.

We conjecture that there exist $\nu_{min}, \nu_{max} \in P$ **satisfying:**

$$\nu_{min} \leq \nu \leq \nu_{max} \; for \; all \; \nu \in P$$

Here, the inequalities denote majorization, which is the standard partial order among partitions:

$$x \leq y \Longrightarrow \sum_{i=1}^{k} x_i \leq \sum_{i=1}^{k} y_i \text{ for all k.}$$

Our Progress

Our *Strengthened* Main Conjecture

We chose to examine the $\nu \leq \nu_{max}$ half of the conjecture in our research.

- Notice that being higher in the partial order corresponds to having more small elements, but overall, this is a **global** condition.
- Through a large-scale computer search, we have found that for every example we checked, the shuffle tableau T_{max} corresponding to the upper bound is **not only unique but is also minimal in *every element***.
- Trivially, if this new conjecture is true, then it solves the above half of the original conjecture.

Our *Strengthened* Main Conjecture

We chose to examine the $\nu \leq \nu_{max}$ half of the conjecture in our research.

- This improved conjecture motivates a ***purely local*** approach to the problem.
- In particular, we examine specific cases where we can decrease an element by one unit based on the value of its neighbors without changing the Temperley-Lieb type.
    - Under the conjecture that every element is itself minimal, we can iteratively minimize various parts of our shuffle tableaux.
- In particular, by extending this method to a complete **set of local rules**, it may be possible to prove that these rules are sufficient to uniquely determine the coefficients of T_{max}. We leave this to the next generation.

Our *Strengthened* Main Conjecture

We chose to examine the $\nu \leq \nu_{max}$ half of the conjecture in our research.

Notice that being higher in the partial order corresponds to having more small elements, but overall, this is a **global** condition. Through a large-scale computer search, we have found that for every example we checked, the tableau T_{max} corresponding to the upper bound is **not only unique but is also minimal in *every element***. Trivially, if this new conjecture is true, then it solves the above half of the original conjecture.

This improved conjecture motivates a ***purely local*** approach to the problem. In particular, we examine specific cases where we can decrease an element by one unit based on the value of its neighbors without changing the Temperley-Lieb type.

In particular, by extending this method to a complete **set of local rules**, it may be possible to prove that these rules are sufficient to uniquely determine the coefficients of T_{max}. We leave this to the next generation.

Local Rules

Again, we used a large-scale computer search.

Our code appears to indicate that an element falling into one of the following cases is necessary (but not sufficient) for it to be able to be decremented without changing the Temperley-Lieb type:

0. All trivial cases.

Local Rules

Again, we used a large-scale computer search.

Our code appears to indicate that an element falling into one of the following cases is necessary (but not sufficient) for it to be able to be decremented without changing the Temperley-Lieb type:

1.

$$
\begin{array}{ccccc}
 & & (A) & - & x-1 \\
(B) & & x & — & (D) \\
| & & | & & \\
x & & (C) & &
\end{array}
\implies
\begin{array}{ccccc}
 & & (A) & & x-1 \\
 & & | & & | \\
(B) & - & x-1 & & (D) \\
 & x & — & (C) &
\end{array}
$$

(Decreasing the central value by 1 is **guaranteed** to preserve the TL type when **both** (C)/(D) and (A)/(B) are connected externally.)

Local Rules

Again, we used a large-scale computer search.

Our code appears to indicate that an element falling into one of the following cases is necessary (but not sufficient) for it to be able to be decremented without changing the Temperley-Lieb type:

2.

$$\begin{array}{ccc} & (A) - \leq x-2 & \\ (B) & x & — (D) \\ | & | & \\ x & (C) & \end{array} \Longrightarrow \begin{array}{c} (A) — \leq x-2 \\ (B) - x-1 — (D) \\ x — (C) \end{array}$$

~~~~~~~or~~~~~~~

$$\begin{array}{c} (A) - x-1 \\ (B) — x — (D) \\ > x - (C) \end{array} \Longrightarrow \begin{array}{cc} (A) & x-1 \\ | & | \\ (B) - x-1 & (D) \\ > x — (C) & \end{array}$$

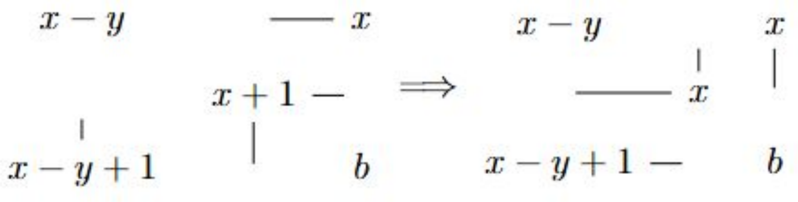
(Here, decreasing the central value by 1 is **guaranteed** to preserve the TL type when **at least one** of (C)/(D) and (A)/(B) are connected externally.)
~~~~~~~

Local Rules

Again, we used a large-scale computer search.

Our code appears to indicate that an element falling into one of the following cases is necessary (but not sufficient) for it to be able to be decremented without changing the Temperley-Lieb type:

3.

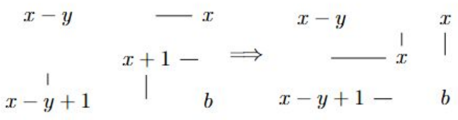


(In this configuration, decreasing the central value by 1 preserves TL type under certain conditions for x,y>0.)

Local Rules

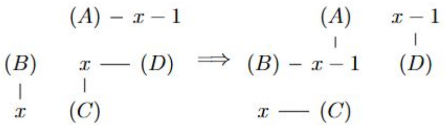
Our code appears to indicate that an element falling into one of the following cases is *necessary* (but not *sufficient*) for it to be able to be decremented without changing the Temperley-Lieb type (besides all trivial cases, including cases where an E_i can be applied, as well as a generalized nonlocal version of the E_i):

(1)



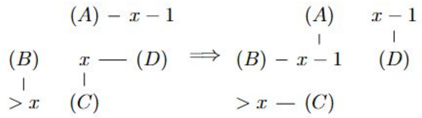
(Decreasing the central value by 1 preserves TL type under certain conditions for x,y>0.)

(2)

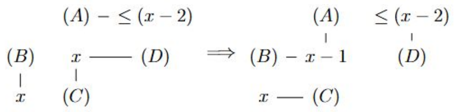


(Decreasing the central value by 1 is guaranteed to preserve the TL type when **both** (C)/(D) and (A)/(B) are connected externally.)

(3)



or



(Here, decreasing the central value by 1 is guaranteed to preserve the TL type when **at least one of** (C)/(D) and (A)/(B) are connected externally.)

Acknowledgements

- First and foremost, thank you to my mother, who taught me to work hard and who was the first person to introduce me to math and physics. I would not be here today without her influence!
- I would also like to thank my mentor Son Nguyen for his large amount of feedback, as well as for helping me obtain the background I needed for this project.
- On a similar note, thank you to Tanya Khovanova, Slava Gerovitch, Pavel Etingof, and Yufei An for making the PRIMES-USA program possible.
- Lastly, thank you to Rachel Chen, Rupert Li, Amogh Akella, and (again) Son Nguyen for proofreading this presentation and giving me suggestions!

Works Cited

[ACM10] Olga Azenhas, Alessandro Conflitti, and Ricardo Mamede. Multiplicity-free skew Schur functions with full interval support. *arXiv preprint arXiv:1009.4170*, 2010.

[Aze99] Olga Azenhas. The admissible interval for the invariant factors of a product of matrices. *Linear and Multilinear Algebra*, 46(1-2):51–99, 1999.

[BB81] Alexandre Beĭlinson and Joseph Bernstein. Localisation de g-modules. *C. R. Acad. Sci. Paris Sér. I Math.*, 292(1):15–18, 1981.

[BFZ96] Arkady Berenstein, Sergey Fomin, and Andrei Zelevinsky. Parametrizations of canonical bases and totally positive matrices. *advances in mathematics*, 122(1):49–149, 1996.

[BH20] Petter Brändén and June Huh. Lorentzian polynomials. *Annals of Mathematics*, 192(3):821–891, 2020.

[BK81] Jean-Luc Brylinski and Masaki Kashiwara. Kazhdan-Lusztig conjecture and holonomic systems. *Inventiones mathematicae*, 64(3):387–410, 1981.

[CCRMM23] Federico Castillo, Yairon Cid-Ruiz, Fatemeh Mohammadi, and Jonathan Montaño. Double Schubert polynomials do have saturated Newton polytopes. In *Forum of Mathematics, Sigma*, volume 11, page e100. Cambridge University Press, 2023.

[Du92] Jie Du. Canonical bases for irreducible representations of quantum GL_n. *Bulletin of the London Mathematical Society*, 24(4):325–334, 1992.

[Du95] Jie Du. Canonical bases for irreducible representations of quantum GL_n, ii. *Journal of the London Mathematical Society*, 51(3):461–470, 1995.

[EH20] Christopher Eur and June Huh. Logarithmic concavity for morphisms of matroids. *Advances in Mathematics*, 367:107094, 2020.

[FMD18] Alex Fink, Karola Mészáros, and Avery St Dizier. Schubert polynomials as integer point transforms of generalized permutahedra. *Advances in Mathematics*, 332:465–475, 2018.

[FZ02] Sergey Fomin and Andrei Zelevinsky. Cluster algebras I: foundations. *Journal of the American mathematical society*, 15(2):497–529, 2002.

[GHS17] Marisa Gaetz, Will Hardt, and Shruthi Sridhar. Support equalities among ribbon Schur functions. *arXiv preprint arXiv:1709.03011*, 2017.

[Hai93] Mark Haiman. Hecke algebra characters and immanant conjectures. *Journal of the American Mathematical Society*, 6(3):569–595, 1993.

[HMMSD22] June Huh, Jacob Matherne, Karola Mészáros, and Avery St Dizier. Logarithmic concavity of Schur and related polynomials. *Transactions of the American Mathematical Society*, 375(6):4411–4427, 2022.

[Kas91] Masaki Kashiwara. On crystal bases of the q-analogue of universal enveloping algebras. *Duke mathematical journal*, 63(2):465–516, 1991.

[Kas93] Masaki Kashiwara. Global crystal bases of quantum groups. *Duke mathematical journal*, 69(2):455–485, 1993.

[LPP07] Thomas Lam, Alexander Postnikov, and Pavlo Pylyavskyy. Schur positivity and Schur log-concavity. *American journal of mathematics*, 129(6):1611–1622, 2007.

[LR34] Dudley Ernest Littlewood and Archibald Read Richardson. Group characters and algebra. *Philosophical Transactions of the Royal Society of London. Series A, Containing Papers of a Mathematical or Physical Character*, 233(721-730):99–141, 1934.

[Lus90] George Lusztig. Canonical bases arising from quantized enveloping algebras. *Journal of the American Mathematical Society*, 3(2):447–498, 1990.

[Lus94] George Lusztig. Total positivity in reductive groups. In *Lie theory and geometry: in honor of Bertram Kostant*, pages 531–568. Springer, 1994.

[Lus97] George Lusztig. Total positivity and canonical bases. *Algebraic groups and Lie groups*, 9:281–295, 1997.

[MTY19] Cara Monical, Neriman Tokcan, and Alexander Yong. Newton polytopes in algebraic combinatorics. *Selecta Mathematica*, 25(5):66, 2019.

[MvW12] Peter RW McNamara and Stephanie van Willigenburg. Maximal supports and Schur-positivity among connected skew shapes. *European Journal of Combinatorics*, 33(6):1190–1206, 2012.

[NNW25] Chau Nguyen, Son Nguyen, and Dora Woodruff. Shuffle Tableaux, Littlewood–Richardson Coefficients, and Schur Log-Concavity. *arXiv preprint arXiv:2506.00349*, 2025.

[NP25] Son Nguyen and Pavlo Pylyavskyy. Temperley–Lieb crystals. *International Mathematics Research Notices*, 2025(7), 2025.

[RS05] Brendon Rhoades and Mark Skandera. Temperley-Lieb immanants. *Annals of Combinatorics*, 9(4):451–494, 2005.

[RS06] Brendon Rhoades and Mark Skandera. Kazhdan–Lusztig immanants and products of matrix minors. *Journal of Algebra*, 304(2):793–811, 2006.

[RSVW07] Victor Reiner, Kristin M Shaw, and Stephanie Van Willigenburg. Coincidences among skew Schur functions. *Advances in Mathematics*, 216(1):118–152, 2007.

[Ska08] Mark Skandera. On the dual canonical and Kazhdan–Lusztig bases and 3412-, 4231-avoiding permutations. *Journal of Pure and Applied Algebra*, 212(5):1086–1104, 2008.