On Polynomial Evaluation at Algebraic Numbers

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Overview

- Certain subsets of $\mathbb{C}$, which are closed under both standard addition and multiplication, are called **semirings**
- $\mathbb{N}_0$ is a semiring, and we can consider its **simple extensions** $\mathbb{N}_0[\alpha]$
- Formally, for $\alpha \in \mathbb{C}$, we set $\mathbb{N}_0[\alpha] := \{f(\alpha) \mid f(x) \in \mathbb{N}_0[x]\}$
  - Here $\mathbb{N}_0[x]$ is the set of polyonomials with nonnegative integer coefficients
- $\mathbb{N}_0[\alpha]$ has been actively studied in recent literature (by Chapman, Gotti, Polo, et al.)

Overview (cont.)

Our Main Algebraic Objects.

$$\mathbb{N}_0[\alpha] := \{f(\alpha) : f(x) \in \mathbb{N}_0[x]\}$$

- An algebraic structure with no irreducibles is called antimatter
- Introduced in 1999 by Coykendall, Dobbs, and Mullins

Goal. Understand the antimatter property for semirings $\mathbb{N}_0[\alpha]$.

The Prototypical Monoid

$\mathbb{N}_0 := \{0, 1, 2, \ldots, 6, 7, \ldots\}$ denotes the nonnegative integers

- Not just a set, but has an operation — addition
  - Commutative, associative, and has identity (0)

Definition. Any set with a binary operation that satisfies the above properties is a **monoid**.

Multiplicative Closure

- $\mathbb{N}_0$ is a special monoid — *closed under multiplication*
  - Commutative, associative, has identity (1), and distributes over addition
- Can we find other monoids closed under multiplication?

Cyclic Extensions

The most natural way to extend $\mathbb{N}_0$ is to introduce an element

- Let $\alpha \in \mathbb{C}$ be any complex number
- Simply taking $\mathbb{N}_0 \cup \{\alpha\}$ might not produce a monoid — it may not be possible to define addition or multiplication
  - Adding one element forces us to add more, like $6\alpha + \alpha^7$
- Instead, we must insert all *finite sums* of powers of α

Formalize with polynomial evaluation at α, like $6x + x^7$

Polynomial Evaluation

Definition. $\mathbb{N}_0[x] :=$ **polynomials** with coefficients in $\mathbb{N}_0$.

- Analogous to $\mathbb{Z}[x]$

To ease notation, we denote $\mathbb{N}_0[\alpha]$ by $M_\alpha := \{p(\alpha) \mid p(x) \in \mathbb{N}_0[x]\}$

Definition. We call M_α the **cyclic monoid** generated by α

Example

When is M_α just the same as $\mathbb{N}_0$?

- If α is not already part of $\mathbb{N}_0$, then M_α strictly contains $\mathbb{N}_0$
- If α is already part of $\mathbb{N}_0$, then adjoining α does nothing

Remark. M_α is the *same monoid* as $\mathbb{N}_0$ if and only if $\alpha \in \mathbb{N}_0$.

Isomorphism

What does it mean for two structures to be (essentially) the same?

- We can relabel their elements so they become indistinguishable
  - Bijection — invertible, i.e., one-to-one and onto
  - Homomorphism — compatible with operations of both monoids

Notation. This equivalence is denoted by $\cong$, read **isomorphic**.

$M_\alpha \cong \mathbb{N}_0$ if and only if $\alpha \in \mathbb{N}_0$

Transcendental Extensions

What is M_π?

- π is *transcendental* — does not interact with anything else
- Similar to $\mathbb{N}_\circ[x]$, with the variable x

Remark. The monoid M_π is **isomorphic** to $\mathbb{N}_\circ[x]$.

Same happens with any transcendental

Algebraic Extensions

Definition. A complex number $\alpha \in \mathbb{C}$ is **algebraic** if α satisfies a nonzero polynomial with rational coefficients.

- Every complex number is either transcendental or algebraic

Convention. Throughout our talk, we tacitly assume α is algebraic.

Correa-Morris and Gotti (2022) provide the first systematic study of the atomicity and factorization of M_α for algebraic α

Minimal Polynomials

- Recall — α is algebraic if it is a root of a polynomial in $\mathbb{Q}[x]$
- Many polynomials have α as a root
  - Restrict to polynomials with *least degree* possible
  - Restrict to *monic* polynomials — leading coefficient is 1

Those two conditions guarantee a unique polynomial $m_\alpha(x) \in \mathbb{Q}[x]$

Definition. This $m_\alpha(x)$ is called the **minimal polynomial** of α.

Algebraic Conjugates

Each α is only associated with one $m_\alpha(x)$

- However, a particular $m_\alpha(x)$ may have many roots

Definition. α and β are **(algebraic) conjugates** if $m_\alpha(x) = m_\beta(x)$.

- **Example.** $\{6+7i, 6-7i\}$, or $\{\sqrt{67}, -\sqrt{67}\}$, or even $\{\sqrt[6]{7}, e^{i\pi/3}\sqrt[6]{7}\}$

Algebraic Conjugates (cont.)

Algebraic conjugates are roots to the same polynomials

Theorem (Correa-Morris and Gotti, 2022)

If α and β are algebraic conjugates, then $M_\alpha \cong M_\beta$.

Recall M_α refers to the additive structure of the semiring $\mathbb{N}_0[\alpha]$

- **Example.** $M_{\sqrt{67}} \cong M_{-\sqrt{67}}$ since $m_{\sqrt{67}}(x) = m_{-\sqrt{67}}(x) = x^2 - 67$

Positive Conjugates

Suppose α has a positive conjugate, say β

- M_β only has nonnegative elements
  - The only element with an *additive inverse* is the identity element 0
- $M_\alpha \cong M_\beta$ also has 0 as the only element with an additive inverse

Suppose α has no positive conjugates

- M_α is an *abelian group* (Gotti, Hong, and Li, 202?)
- *Divisibility* structure is uninteresting

Convention. We take α to be **positive** and **algebraic**.

Atoms

- We can build up $M_{\sqrt{67}}$ with 1 and $\sqrt{67}$
- 1 and $\sqrt{67}$ cannot be built up from other elements themselves

Definition. An element a is an **atom** if a is nonzero, and whenever $b + c = a$, then either b or c is 0.

- **Example.** If $\alpha > 1$, then 1 is an atom in M_α
- **Example.** 7 is never an atom since $1 + 6 = 7$

Atoms are *building blocks*, like primes in the natural numbers

A Monoid with No Atoms

Definition. A monoid is **atomic** if every nonzero element can be decomposed as a sum of atoms.

- **Example.** $M_{1/7}$ is the set of all nonnegative fractions with the denominator a power of seven
  - $a = \dfrac{6a}{7} + \dfrac{a}{7}$ for any a, so a is never an atom
- $M_{1/7}$ has no atoms, so it is not atomic

Antimatterness

Definition. A monoid is **antimatter** if it contains no atoms.

- First studied by Coykendall, Dobbs, and Mullins (1999)

Are atomicity and antimatterness related?

Theorem (Correa-Morris and Gotti, 2022)

The following statements are equivalent for positive (algebraic) α.

(a) M_α is atomic.

(b) M_α is not antimatter.

(c) 1 is an atom of M_α.

Rational Antimatter Monoids

Start with the case where α is rational — take $q \in \mathbb{Q}$ and $q > 0$

> **Proposition.** M_q is antimatter $\iff q^{-1} > 1$ is an **integer**.

- If $q \geqslant 1$, then 1 is the *least non-zero element* (hence, atom)
  - We modify this argument slightly for $q < 1$
- **Example.** Generators in $M_{6/7}$ are 1, $6/7$, $(6/7)^2 = 36/49$, and so on
  - 1 is the *least non-zero element with odd numerator* (hence, atom)
- More generally, 1 is an atom if $q = 1$ or q is not a unit fraction
  - From before, 1 is not an atom if $q < 1$ and q is a unit fraction

Algebraic Integers

M_α can be antimatter even when α is irrational

- If $\alpha = 1/\sqrt[6]{7}$, then $\alpha^6 = 1/7$, so M_α is still antimatter

Lemma. If M_α is antimatter, then α^{-1} is an **algebraic integer**.

- Algebraic integers generalize the definition of the rational integers
  - 67 is an integer, and $m_{67}(x) = x - 67$
  - 6.7 is not an integer, and $m_{6.7}(x) = x - 6.7$
- α is an **algebraic integer** if $m_\alpha(x) \in \mathbb{Z}[x]$

An Irrational Example

Example. For φ the golden ratio, $M_{\varphi^{-1}}$ is antimatter.

- $m_\varphi(x) = x^2 - x - 1 \in \mathbb{Z}[x]$, which makes φ an algebraic integer
- Define $\alpha := \varphi^{-1}$, and manipulate to get $1 = \alpha + \alpha^2$
  - Hence $\alpha^n = \alpha^{n+1} + \alpha^{n+2}$ for every $n \in \mathbb{N}_0$
  - Indeed, for any $p(\alpha) \in M_\alpha$, then $p(\alpha) = p(\alpha)\alpha + p(\alpha)\alpha^2$
- Every element is a sum of smaller positive elements, so no atoms
- $M_{\varphi^{-1}}$ is antimatter

Antimatter Decomposition

Definition. For a given α, an **antimatter decomposition** of 1 is a polynomial $h(x) \in \mathbb{Z}[x]$ with a root at α such that each coefficient is nonnegative except for the constant term of -1.

Crucial part of $M_{\varphi^{-1}}$ was decomposing 1 as the sum of powers of φ^{-1}

- Recall $1 = (\varphi^{-1})^2 + \varphi^{-1}$
- Equivalently, φ^{-1} is a root to $h(x) = x^2 + x - 1$

Decomposition may not exist even when α^{-1} is an algebraic integer

Algebraic Integer Counterexample (cont.)

Proposition. If M_α is antimatter, then $m_\alpha(x)$ has one positive root.

Suppose $h(x) \in \mathbb{Z}[x]$ is any antimatter decomposition

- Aside from the constant, each coefficient is nonnegative
  - ax^n is increasing when $a, n \geqslant 0$, meaning $h(x)$ as a whole is increasing
- $h(x)$ has precisely one positive root by monotonicity
- Each root to $m_\alpha(x)$ is also a root to $h(x)$

Exact Characterization

The antimatter decomposition allows us to prove one more condition about the relative magnitudes of roots.

Theorem (Chen, Gotti, Lu, and Yao, 2025)

If $\alpha \in (0, 1)$ is algebraic, then M_α is antimatter if and only if

- α has no positive conjugate aside from itself,
- α^{-1} is an algebraic integer, and
- $|\rho| \leqslant \alpha^{-1}$ for every conjugate ρ to α^{-1}.

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