Automated Discovery of Extremal Unit-Distance Graphs

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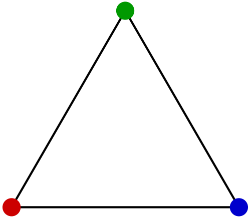
The Hadwiger-Nelson Problem

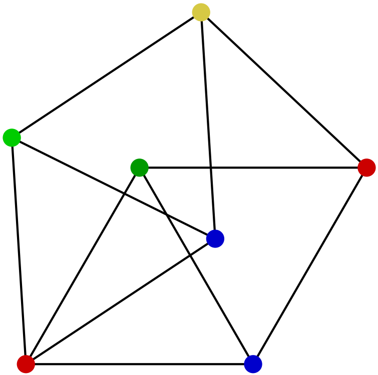
What is the minimum number of colors c required to color each point in the plane such that no two points at a distance 1 have the same color?

Asked by Nelson in 1950.

The Hadwiger–Nelson Problem: $c \geq 3$

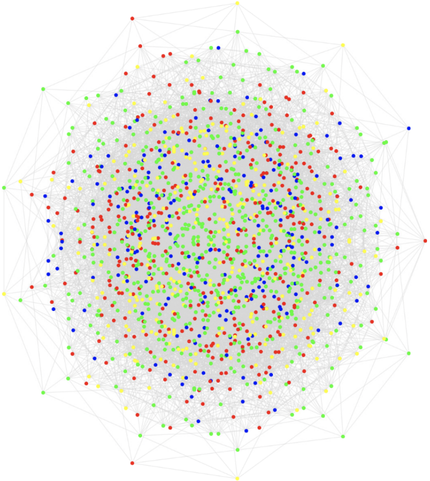


The Hadwiger–Nelson Problem: $c \geq 4$



Moser Spindle

The Hadwiger–Nelson Problem: $c \geq 5$



De Grey Graph (2018), 1581 vertices.

An Equivalent Question

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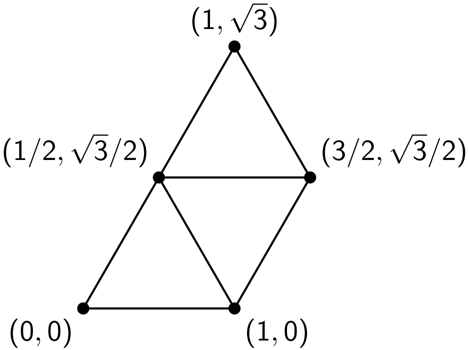
- Let a *unit-distance graph* (UDG) be a graph with vertices being points in the plane, and edges between points with distance 1.
- Over all finite UDGs, what is the maximum chromatic number?



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- How many edges can UDGs in $\mathbb{R}^d$ have, for $d \geq 2$?
- How many edges can UDGs in K^d have, for a number field K?
- **Dense UDGs give us information about many types of extremal UDGs.**

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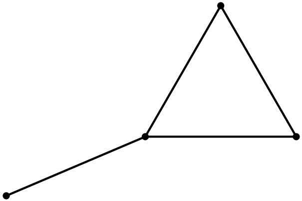
- For $d = 2$, the densest UDGs on up to 21 vertices are known. It is suspected that we have discovered the densest possible UDGs on up to ≈ 30 vertices.
- For $d = 3$, little to no work has been done on computational discovery. Complexity increases greatly.

Searching For a UDG



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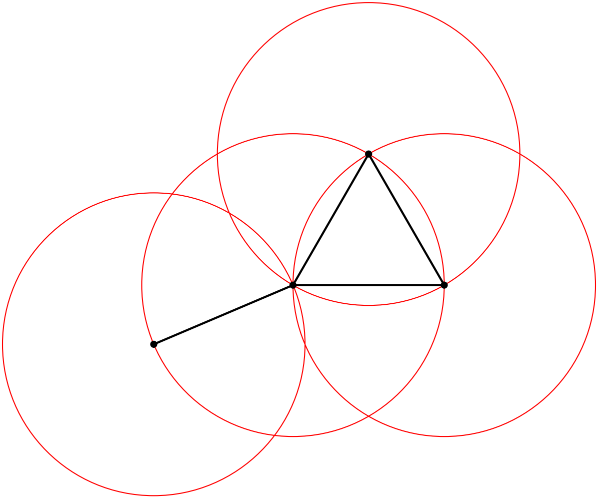

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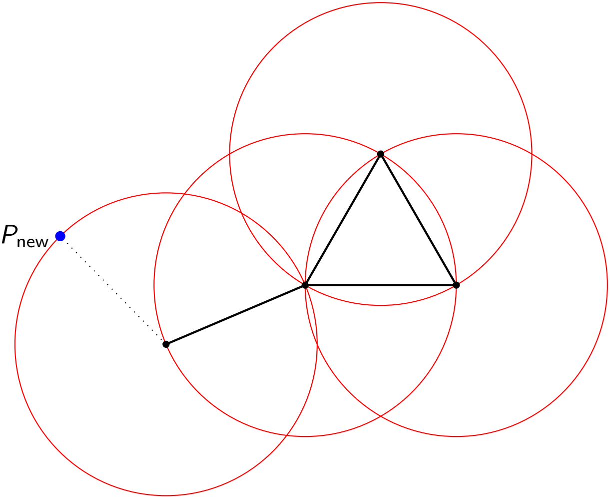

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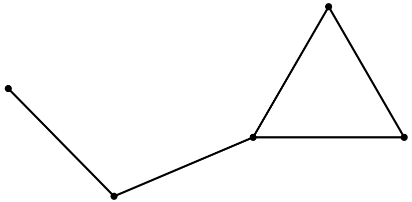


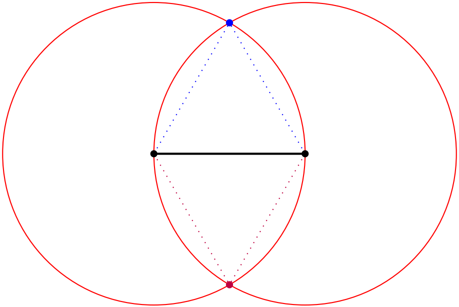
The Two Approaches

- **Continuous**: Use the whole circle.
- **Discrete**: Only use a (carefully selected) finite subset of each circle.

The Continuous Approach

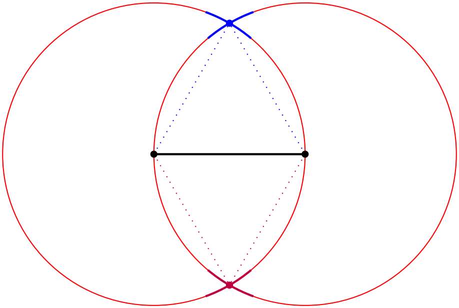
The Continuous Approach: The Issue

Infinite action space $\implies$ probability 0 of anything "useful"



The Continuous Approach: Potential Fix

Relax definition to ε-**UDGs**: include edges between points at distance 1 with a tolerance ε. In other words, edges are drawn between points with distance in $[1-\varepsilon, 1+\varepsilon]$.



The Continuous Approach: Another Issue

The following is an ε-UDG but not a UDG.

The Continuous Approach: A Final Fix

Let a (ε, δ)-**UDG** be an ε-UDG such that no two vertices are within a distance δ from each other.

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Theorem (Aggarwal, 2025)

There exist functions $\varepsilon(n, d), \delta(n, d) > 0$ such that any $(\varepsilon(n, d), \delta(n, d))$ unit-distance graph in $\mathbb{R}^d$ on n vertices is a unit-distance graph.

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The ε given by the proof of the above theorem for a fixed δ is doubly exponentially small (of the form $r^{-s^{\mathrm{poly}(n)}}$), but we conjecture that the tightest possible bound is $O_\delta(n^{-2})$.

The Discrete Approach

The Discrete Approach: Main Idea

Theorem (Aggarwal, 2025)

Let G be a finite unit-distance graph G in $\mathbb{R}^d$. Then there exists a number field $K \subset \mathbb{R}$ such that there exists a finite unit-distance graph G' in K^d such that G is a subgraph of G'.

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Let $K \subset \mathbb{R}$ be a number field which is finite over $\mathbb{Q}$. Let $m \in \mathbb{N}$ and O_K be the ring of integers of K. Then there are finitely many vectors $\mathbf{v} \in \left(\frac{1}{m} \cdot O_K\right)^2$ such that $||\mathbf{v}|| = 1$.

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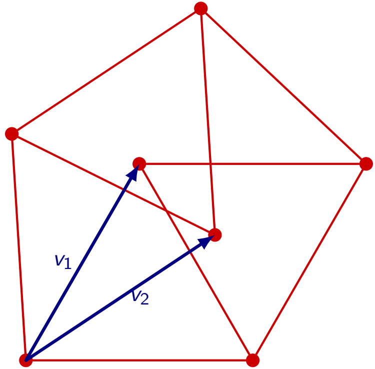
This means that we may select a number field K and a denominator m to search in, and the action space is finite. Furthermore, we have an algorithm to compute it. **In other words, it suffices to look at lattices.**

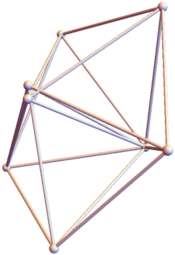
The Discrete Approach: The Moser Lattice

Over $\mathbb{C}$, the lattice generated by the Moser Spindle is $\mathbb{Z}\left[\frac{1+i\sqrt{3}}{2}, \frac{5+i\sqrt{11}}{6}\right]$:



The densest known UDGs in $\mathbb{R}^2$ all lie in the Moser Lattice.

The Discrete Approach: The Raiskii Spindle



Raiskii Spindle in $\mathbb{R}^3$

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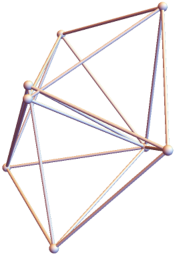

Raiskii Spindle in $\mathbb{R}^3$

Theorem (Aggarwal, 2025)

The lattice generated by the $\mathbb{R}^d$ Raiskii Spindle is a subset of $\mathbb{Q}[\sqrt{7d^2+8d}, \sqrt{2}, \sqrt{d+1}]^d$ and is generated by at most $2(d-1)$ vectors.

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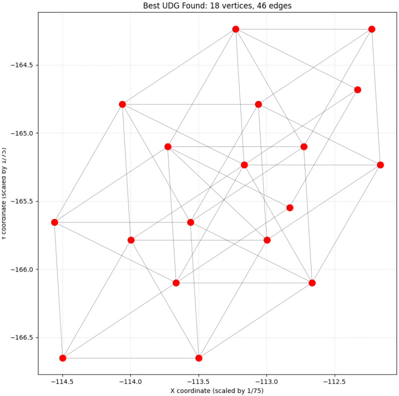
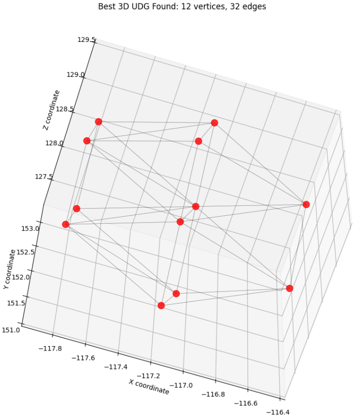
- Results produced **without GPU use** (6-core 2019 Macbook Pro), significantly faster than previous results.
- Developing upper bounds in $\mathbb{R}^3$ is tedious and computationally intensive. It is a work in progress.

Computational Results: Examples



Densest 18 vertex UDG in $\mathbb{R}^2$



Densest 12 vertex UDG in $\mathbb{R}^3$

From Geometry to Quantum Logic

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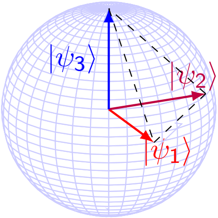
- Specifically, they model relationships between **quantum measurements** that can or cannot be simultaneously assigned definite outcomes.



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  - 18-vector set in $\mathbb{R}^4$ (Cabello et al., 1996)

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- Computational discovery of Kochen-Specker Sets or orthogonality graphs with high contextuality measures is the next step.

Thanks to:

- My wonderful mentors Andrew Gritsevskiy and Dr. Jesse Geneson,
- The MIT PRIMES-USA program for this opportunity to conduct research,
- My family and friends for their support throughout this process.

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