Extending CC0 Circuit Upper Bounds Beyond Symmetric Functions

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Outline

Motivation

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Consider a Boolean function f. We call the set $\{x : f(x) = 1\}$ a language/decision problem associated with f.

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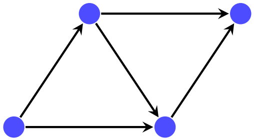
- It is of interest to us to look at complexity classes of efficiently computable languages, i.e. sets of languages that can be computed within a certain resource constraint.
- Circuits are mathematically simpler and are much more powerful than classical computation models, like Turing Machines.

Graphs

Definition (Directed Graph)

A *directed graph* is an ordered pair $G = (V, E)$, where V is a finite set of vertices, and $E \subseteq V \times V$ is a set of ordered pairs (u, v) called directed edges, where each edge goes from vertex u to vertex v.



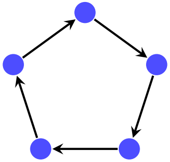
Graphs

Definition (Cycle in a Directed Graph)

A *cycle* in a directed graph is a sequence of distinct vertices

$$v_1, v_2, \ldots, v_k \quad (k \geq 2)$$

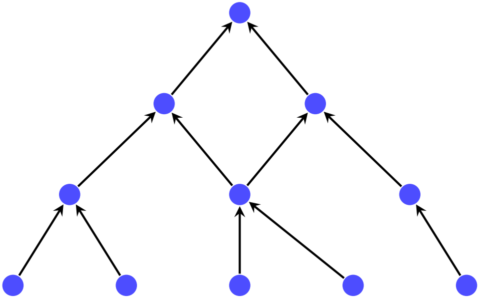
such that $(v_i, v_{i+1}) \in E$ for all $1 \leq i < k$, and additionally $(v_k, v_1) \in E$.



Graphs

Definition (Directed Acyclic Graph)

A *directed acyclic graph* is a directed graph that contains no directed cycles.



What is a circuit?

Definition

A *circuit* is a directed acyclic graph in which nodes of in-degree 0 are called *inputs* and all other nodes are called *gates*. Gates of out-degree 0 are called *outputs*. Each gate g of in-degree k is labeled with a k-ary function f_g, and computes it in a natural way.

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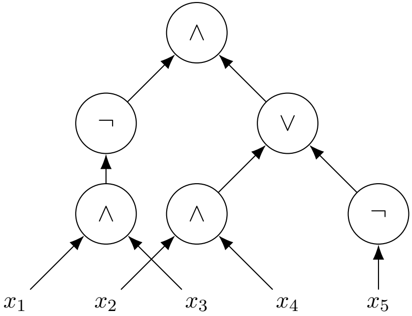

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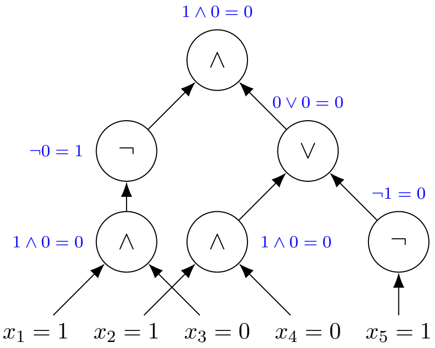
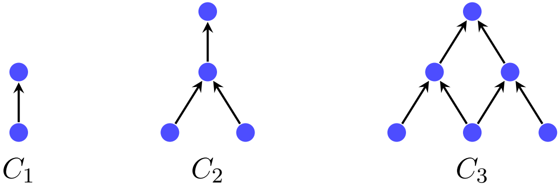


Circuit families

Definition

A *circuit family* is a sequence $\{C_n\}_{n\in\mathbb{N}}$ of circuits, where C_n has n inputs and a single output.



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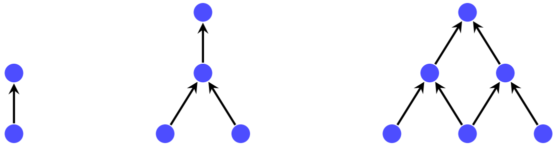

C_1 $|C_1| = 1$

C_2 $|C_2| = 2$

C_3 $|C_3| = 3$

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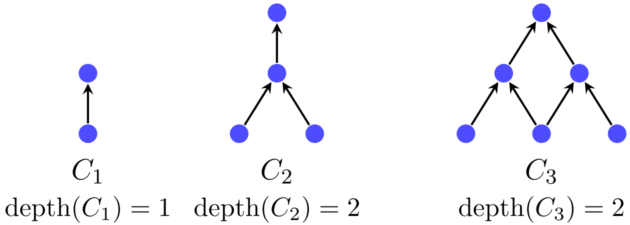

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Relevant Boolean functions

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Definition

A Boolean function f is called *symmetric* if there exists $g : \mathbb{N} \to \{0,1\}$ such that for each binary vector $\mathbf{x}$,

$$f(\mathbf{x}) = g\left(|\mathbf{x}|_1\right),$$

i.e., it depends only on the number of 1s in the input.

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A *counting circuit* is a circuit in which every gate computes a MOD function of unbounded in-degree.

- $\mathsf{CC}^i[m]$ is the class of languages computable by counting circuit families of polynomial size and depth $O(\log^i n)$, where every gate computes the MOD_m function.

Circuit Complexity Classes (cont.)

Definition

An *alternating circuit with counting* is a circuit in which every gate computes either a MOD function, the AND function, or the OR function, all of unbounded in-degree.

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- $\mathsf{AC}^i[m]$ is the class of languages computable by alternating counting circuit families of polynomial size and depth $O(\log^i n)$, where every MOD gate computes the MOD_m function.
- We will focus on CC^0.

Outline

Key lower bound

Theorem (Razborov, Smolensky)

For distinct primes $p \neq q$, any depth-d $\mathrm{AC}^0[q]$ circuit that computes MOD_p must have size at least $\exp\big(\Omega(n^{1/(2d)})\big)$.

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- This is a powerful lower bound which shows the weakness of prime-modulus counting circuits.
- Motivates us to consider composite moduli.

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Theorem (Chapman, Williams)

For every $\varepsilon > 0$, there is a modulus $m \leq (1/\varepsilon)^{2/\varepsilon}$ such that every symmetric function on n bits can be computed by depth-3 MOD_m circuits of $\exp(O(n^\varepsilon))$ size.

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Example

Every symmetric Boolean function on n variables has depth-three MOD_{30} circuits of size $\exp(O(n^{1/3}\log n))$.

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Generalization

- Symmetric functions are those that depend only on the polynomial $\sum_{i=1}^{n} x_i$. If we have a function that is instead dependent only on the value of a different polynomial, we can compute it in a similar manner.

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Theorem

Let f be a Boolean function on n bits $x_1, \ldots, x_n$ and m be a product of $k \geq 2$ distinct primes. Let $I(x) \in \mathbb{Z}[x_1, \ldots, x_n]$ of degree d, whose maximum value over all Boolean inputs is M. Then, every f that depends solely on the value of $I(x)$ can be computed by depth-3 MOD_m circuits of size $\exp(O(d \cdot M^{1/k} \log M))$.

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- What is this function doing? How can we mimic this with a simple polynomial?
- We can use the degree-two polynomial $I(x) = \sum_{i=1}^{n} x_i x_{i+1}$ to give us depth-three MOD_m circuits of size $\exp(O(n^{1/k} \log n))$ computing f.

Symmetry Composition

Theorem

Let $m = p_1 \cdots p_k$ be a product of k distinct primes, and let g be an n-variate symmetric function. For each $1 \leq i \leq n$, let h_i be a function that depends only on n^ϵ inputs. Then $f(\mathbf{x}) := g(h_1(\mathbf{x}), \ldots, h_n(\mathbf{x}))$ can be computed with depth-three MOD_m circuits of size $\exp(O(n^{2/k+\epsilon} \log m))$.

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Corollary

Let g be an n-variate symmetric function. For each $1 \le i \le n$, let h_i be a function that depends only on a subpolynomial number of inputs. Then $f(\mathbf{x}) := g(h_1(\mathbf{x}), \ldots, h_n(\mathbf{x}))$ can be computed with a counting circuit family of depth three and subexponential size.

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Outline

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