Efficient Enumeration of Quadratic Lattices

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Outline for Talk

Background

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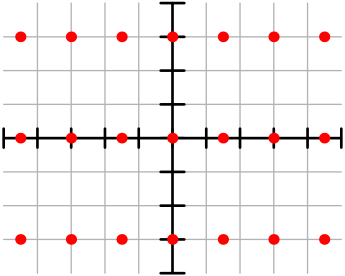

Definition

A quadratic form on $\mathbb{Q}^n$ is a function $P : \mathbb{Q}^n \to \mathbb{Q}$ of the form

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Given P and a choice of a basis for $\mathbb{Q}^n$, there exists a symmetric $n \times n$ matrix M such that

$$2P\left(\begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}\right) = \begin{bmatrix} u_1 \cdots u_n \end{bmatrix} \begin{bmatrix} M \end{bmatrix} \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}.$$

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Definition

Given a quadratic form P and a lattice L, a Gram matrix of L is a matrix M of the form above, where the basis chosen is one that spans L. The determinant of L is the determinant of any one of its Gram matrices.

Background

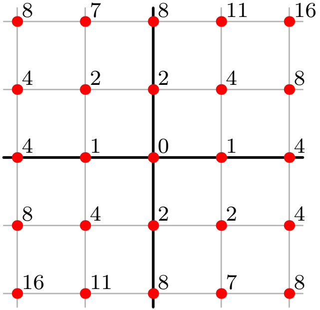
Lattice: $\mathbb{Z}^2$ Quadratic form: $P(x, y) = x^2 + xy + 2y^2$

Choice of basis: $\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}$

Gram matrix: $\begin{bmatrix}2 & 1\\1 & 4\end{bmatrix}$

Determinant: 7

Example: $2P(1,2) = \begin{bmatrix}1 & 2\end{bmatrix}\begin{bmatrix}2 & 1\\1 & 4\end{bmatrix}\begin{bmatrix}1\\2\end{bmatrix} = 22$



Background

Lattice: $\frac{\mathbb{Z}}{2} \times \mathbb{Z}$ Quadratic form: $P(x, y) = 4x^2 + 2xy + 2y^2$

Choice of basis: $\begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

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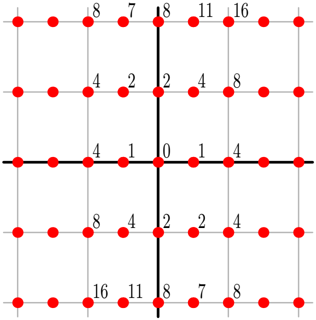

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Given a quadratic form P and a lattice L, we say L is integral if all the entries in the Gram matrix are integers. We say L is even-integral if all the entries in the Gram matrix are integers and the entries on the diagonal are even.

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Example

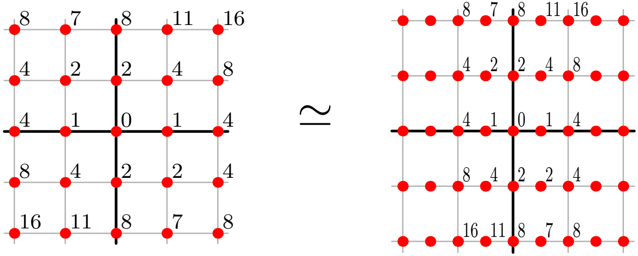
A lattice with gram matrix

$$\begin{pmatrix} 2 & 1 & 0 \\ 1 & 0 & -4 \\ 0 & -4 & 3 \end{pmatrix}$$

is integral, but not even, since 3 is on the diagonal.

Definition

Let L_1 be a lattice with quadratic form P_1 and let L_2 be a lattice with quadratic form P_2. Lattices L_1 and L_2 are isometric if there exists a linear bijection $f : L_1 \to L_2$ such that $P_1(v_1) = P_2(f(v_1))$ for all $v_1 \in L_1$.



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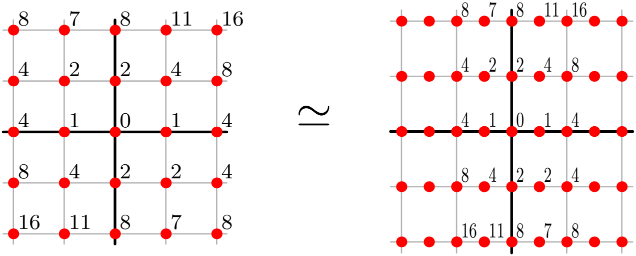

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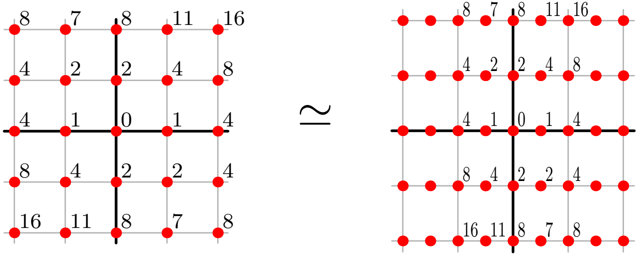

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Theorem (Minkowski)

For a fixed rank and determinant, there are finitely many isometry classes.

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Definition

Let L_1 and L_2 be integral lattices with quadratic forms P_1 and P_2, respectively. Lattices L_1 and L_2 are locally isometric at p if

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Let L_1 and L_2 have gram matrices G_1 and G_2, respectively. An equivalent definition is that for each positive integer k, there exists a matrix $U \in \mathsf{GL}_n(\mathbb{Z}/p^k\mathbb{Z})$ such that

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Equivalently, there exists an invertible $n \times n$ matrix U of reals such that $U^\intercal G_1 U = G_2$.

Definition

The genus of an integral lattice L is the set of all integral lattices L' such that:

- L and L' are locally isometric at p for all primes p,
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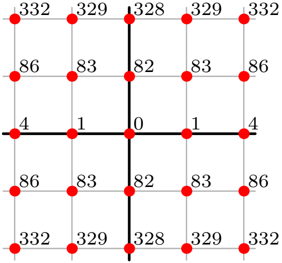
Idea: The genus groups together lattices that "look the same" locally everywhere, though they may differ in actuality.

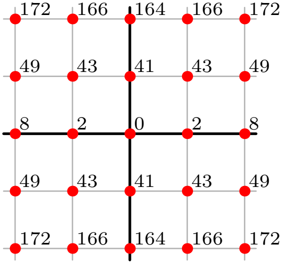
Background

$$L_1 = L_2 = \mathbb{Z}^2, \qquad P_1(x, y) = x^2 + 82y^2, \qquad P_2(x, y) = 2x^2 + 41y^2.$$

Claim: Lattices L_1 and L_2 are in the same genus.



Lattice L_1 with quadratic form P_1



Lattice L_2 with quadratic form P_2

L_1 and L_2 aren't isometric because L_1 represents 1 at $(1, 0)$ while L_2 doesn't represent 1.

Background

$$L_1 = L_2 = \mathbb{Z}^2, \qquad P_1(x,y) = x^2 + 82y^2, \qquad P_2(x,y) = 2x^2 + 41y^2.$$

Theorem

Lattices L_1 and L_2 are in the same genus iff they share a common determinant d and are locally isometric at p for $p \mid 2d$.

- $L_1 \otimes \mathbb{Z}_2 \simeq L_2 \otimes \mathbb{Z}_2$ because we can take the bijection that sends (x, y) to $(y\sqrt{41}, x/\sqrt{41})$; we have

$$P_1(x,y) = P_2\big(y\sqrt{41}, x/\sqrt{41}\big).$$

- Similarly, $L_1 \otimes \mathbb{Z}_{41} \simeq L_2 \otimes \mathbb{Z}_{41}$ because we can take the bijection that sends (x, y) to $(x/\sqrt{2}, y\sqrt{2})$; we have

$$P_1(x,y) = P_2\big(x/\sqrt{2}, y\sqrt{2}\big).$$

- Lastly, $L_1 \otimes \mathbb{R} \simeq L_2 \otimes \mathbb{R}$ because we send (x, y) to $(x/\sqrt{2}, y\sqrt{2})$.

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A genus symbol is a compact way to encode the local invariants of a quadratic lattice at all primes p. It records the isometry class of $L \otimes \mathbb{Z}_p$ for each p.

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Example

The genus symbol of $L = \mathbb{Z}^2$ and $P(x, y) = x^2 + 82y^2$ tells us that the quadratic form is positive definite and the determinant is 82. It also gives us additional info on $p = 2$ and $p = 41$. For example, at $p = 41$ the local genus symbol is

$$1^1 41^1$$

Problem

Problem: Given a rank n and determinant d, find every genus of that rank and determinant and find a representative for each genus (a lattice in the genus).

Counting Genus Symbols

To find the complexity of finding a representative for every genus of a given rank and determinant, we need to know how many genus symbols have that rank and determinant.

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Theorem (A.-C.-G.-W., 2025)

Let $n, d \in \mathbb{Z}$ such that $n > 0$. Let $p \mid d$ be an odd prime with $\nu_p(d) < n$. Then $L_p(n, d)$, which is the number of valid p-adic genus symbols of rank n and determinant d is

$$L_p(n, d) \sim \frac{1}{8\nu_p(d)} e^{\pi\sqrt{\nu_p(d)}}.$$

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However, for $p = 2$, it is significantly harder to count the number of genus symbols due to the inability to diagonalize the form. Therefore, a precise formula for the number of genus symbols for $p = 2$ is difficult.

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Theorem (A.-C.-G.-W., 2025)

The Brandhorst Algorithm has the same time complexity as finding a maximal lattice.

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Corollary (A.-C.-G.-W., 2025)

The Brandhorst Algorithm for computing representatives runs in $O(2^{4n} + n^{1+\psi} \log d)$ time.

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Theorem (A.-C.-G.-W., 2025)

The Hanke algorithm runs in $O(2^n n^3)$ time.

This algorithm still runs in exponential time, but the cost of this exponential step is faster than the exponential step in the previous algorithm.

However, we are able to optimize the algorithm. The only exponential step is finding an even 2-neighbor of our current lattice. A 2-neighbor of a lattice L is a lattice L' such that $L/(L \cap L') \cong L'/(L \cap L') \cong \mathbb{F}_2$.

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- It is known in literature that such a vector always exists for quadratic forms of at least 5 variables. Therefore, we can set any variable after the 5th to zero, and brute force the remaining up to 8^5 possibilities.
- This allows the algorithm to be optimized to $O(n^3 \log d)$, which is the fastest known algorithm to compute maximal overlattices.

- Using the Hanke algorithm, we can achieve a complexity of $O(n^3 \log d)$; faster than another existing $O(n^8 \log d)$ algorithm for computing representatives (due to Dubey and Holenstein).

Conclusion and Future Steps

- Using the Hanke algorithm, we can achieve a complexity of $O(n^3 \log d)$; faster than another existing $O(n^8 \log d)$ algorithm for computing representatives (due to Dubey and Holenstein).
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- We hope to implement and test these algorithms to classify more lattices on the LMFDB, which is a database of number theoretic objects.
- We can use the local version of the optimized Hanke algorithm in the Sage algorithm to speed it up.

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